Blow Ups of Complex Solutions of the $3\mathcal{D}$-Navier-Stokes System and Renormalization Group Method

by

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Abstract: We consider complex-valued solutions of the three-dimensional Navier-Stokes system without external forcing on R^3. We show that there exists an open set in the space of 10-parameter families of initial conditions such that for each family from this set there are values of parameters for which the solution develops blow up in finite time.

Keywords: Navier-Stokes system, renormalization group theory, fixed point, the linearization near the fixed point, spectrum of the linearized group, Hermite polynomials

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§1. Introduction.

There are many phenomena in nature which can be considered as some manifestation of blow ups, like hurricanes, tornadoes, sandstorms, etc. If we believe that Navier-Stokes system describes well enough the motions of real gases and fluids under normal conditions, then it gives some reasons to expect that blow ups in solutions of this system also exist.

We consider in this paper the $3\mathcal{D}$-Navier-Stokes system for incompressible fluids moving without external forcing on R^3 with viscosity equal to 1. After Fourier transform it becomes a non-local, non-linear equation for a non-known function $v(k,t)$ with values in C^3, $k \in R^3$, $t > 0$. The incompressibility condition takes the form* $\langle v(k,t),\, k\rangle = 0$ and

$$v(k,t) = \exp\{-t|k|^2\}\, v(k,0) + i\int_0^t \exp\{-(t-s)|k|^2\}\, ds \cdot \int_{R^3} \langle\, v(k-k',s),\, k\rangle \cdot P_k\, v(k',s)\, dk' \tag{1}$$

In this expression $v(k,0)$ is an initial condition and P_k is the orthogonal projection to the subspace orthogonal to k, i.e. $P_k v = v - \frac{\langle v,k\rangle \cdot k}{\langle k,k\rangle}$. The formula (1) shows that the Navier-Stokes system is genuinely infinite-dimensional dynamical system: the value $v(k,t)$ is determined by the integration over all "degrees of freedom" and previous moments of time.

The problem of blow ups in solutions of the Navier-Stokes System(NSS) appeared after classical works of J. Leray (see [Le 1]) where he proved the existence of the weak solutions of NSS. O. Ladyzenskaya proved the existence of strong solutions of 2-dim NSS in bounded domains (see [La 1]). Many important contributions to the modern understanding of the 2-dim fluid dynamics were done by C. Foias and R. Temam (see [FT]), V. Yudovich (see [Y1]), Giga ([G1]) and others. However, the situation with the 3-dim NSS remained unclear. The Clay mathematical institute announced the problem of existence of strong solutions of the 3-dimensional NSS as one of the most important problem in mathematics of the XXI-century (see [Cl]).

In this paper we omit the condition that $v(k,t)$ is the Fourier transform of a real-valued vector field in the x-space and consider (1) in the space of all possible complex-valued functions with values in C^3. In this situation the energy inequality does not hold. Detailed

*Since $k \in R^3$, $v(k,t) \in C^3$, the order in the inner product is important.

assumptions concerning the initial condition $v(k,0)$ will be discussed later (see §7). In all cases $v(k,0)$ will be bounded functions whose support is a neighborhood of some point $(0,0,k^{(0)})$. The incompressibility condition implies that the components $v_1(k,0)$,$v_2(k,0)$ of $v(k,0)$ are arbitrary functions of k while $v_3(k,0)$ can be found from the incompressibility condition $\langle v,k\rangle = 0$.

Various methods (see, for example, [K], [C], [S1]) allow to prove in such cases the existence and uniqueness of classical solutions of (1) on finite intervals of time. For these solutions (see, for example [S2])

$$|v(k,t)| \leq \mathsf{const}\, \exp\left\{-\,\mathsf{const}\,\sqrt{t}\,\cdot\,|k|\right\},\, 0 \leq t \leq t_0\,. \tag{2}$$

Presumably, $v(k,t)$ has an asymptotics of this type but this requires more work. According to a conventional wisdom, possible blow ups are connected with the violation of (2).

In this paper we fix t and consider one-parameter families of initial conditions $v_A(k,t) = Av(k,0)$. We show that for some special $v(k,0)$ one can find critical values $A_{cr} = A_{cr}(t)$ such that the solution $v_{A_{cr}}(k,s)$ blows up at t so that for $t' < t$ both the energy and the enstrophy are finite while at $t' = t$ they both become infinite. Even more, for $t' < t$ the solution decays exponentially outside some region depending on t. As $t' \uparrow t$ this region expands to an unbounded domain in R^3.

Our main approach is based on the renormalization group method which is so useful in probability theory, statistical physics and the theory of dynamical systems. It is rather difficult to give the exact formulation of our result in the introduction because it uses some notions, parameters, etc., which will appear in the later sections. Loosely speaking, we show that in ℓ-parameter families of initial conditions, for $\ell = 10$, one can find values of parameters for which the solutions develop blow ups of the type we already described. The meaning of ℓ is explained in §4, §5, §6.

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§2. Power Series for Solutions of the $3\mathcal{D}$-Navier-Stokes-Systems and Preliminary Changes of Variables

Our general approach is based upon the method of power series which were introduced in [S1], [S2]. We write down the solution of (1) in the form:

$$v_A(k,t) \,=\, \exp\{-t|k|^2\} \,\cdot\, A\, v(k,0) \,+\, \int\limits_0^t \exp\{-(t-s)|k|^2\} \,\cdot\, \sum_{p>1} A^p\, h_p(k,s)\, ds \qquad (3)$$

The substitution of (3) into (1) gives the system of recurrent equations connecting the functions h_p:

$$h_1(k,s) \,=\, \exp\{-s|k|^2\}\, v(k,0), \qquad (4)$$

$$h_2(k,s) \,=\, i \int\limits_{R^3} \langle v(k-k',0)\,,k\,\rangle\, P_k\, v(k',0) \,\cdot\, \exp\{-s|k-k'|^2 \,-\, s|k'|^2\}\, d^3k', \qquad (5)$$

$$\begin{aligned} h_p(k,s) \,=\, & i\int_0^s ds_2 \int\limits_{R^3} \langle\, v(k-k',0),k\,\rangle\; P_k h_{p-1}(k',s_2) \,\cdot \\ & \exp\{-s|k-k'|^2-(s-s_2)|k'|^2\}\, d^3k' \,+\, i \sum_{\substack{p_1+p_2=p \\ p_1,p_2>1}} \int_0^s ds_1 \int_0^s ds_2 \int\limits_{R^3} \langle\, h_{p_1}(k-k',s_1),k\,\rangle \,\cdot \\ & P_k h_{p_2}(k',s_2) \,\cdot\, \exp\{-(s-s_1)|k-k'|^2-(s-s_2)|k'|^2\}\, d^3k' \,+ \\ & i\int_0^s ds_1 \int\limits_{R^3} \langle\, h_{p-1}(k-k',s_1),\, k\,\rangle\; P_k\, v(k',0) \,\cdot\, \exp\{-(s-s_1)|k-k'|^2 - s|k'|^2\}\, d^3k'\,. \end{aligned} \qquad (6)$$

Clearly, $h_p(k,s) \,\perp\, k$ for every $p \geq 1$, $k \in R^3$.

It follows from the results of [S2] that the series (3) converges for sufficiently small s and gives a classical solution of (1). Make the following change of variables which simplifies (4), (5), (6). Put $\tilde{k} \,=\, k\sqrt{s}$, $\tilde{k}' \,=\, k'\sqrt{s}$, introduce relative times $\tilde{s}_1, \tilde{s}_2, s_1 \,=\, \tilde{s}_1 s, s_2 \,=\, \tilde{s}_2 s$ and denote $g_r(\tilde{k},s) = h_r\left(\frac{\tilde{k}}{\sqrt{s}},s\right), r \geq 1$. Then

$$g_1(\tilde{k},s) \,=\, \exp\{-|\tilde{k}|^2\} \,\cdot\, v\left(\frac{\tilde{k}}{\sqrt{s}},0\right)\,, \qquad (4')$$

$$g_2(\tilde{k},s) \;=\; h_2\left(\frac{\tilde{k}}{\sqrt{s}},s\right) \;=\; \frac{i}{s^2}\int\limits_{R^3} \langle\, v\left(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}},0\right),\tilde{k}\,\rangle \cdot$$

$$P_{\tilde{k}} v\left(\frac{\tilde{k}'}{\sqrt{s}},0\right)\,\exp\{-|\tilde{k}-\tilde{k}'|^2-|\tilde{k}'|^2\}\,d^3\tilde{k}'\,, \tag{5'}$$

$$g_p(\tilde{k},s) \;=\; \frac{i}{s}\int\limits_0^1 d\tilde{s}_2\int\limits_{R^3}\langle\, v\left(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}},0\right),\tilde{k}\,\rangle\,\cdot P_{\tilde{k}}\, g_{p-1}(\tilde{k}\sqrt{\tilde{s}_2}\,,\tilde{s}_2 s)$$

$$\exp\{-|\tilde{k}-\tilde{k}'|^2-(1-\tilde{s}_2)|\tilde{k}'|^2\}d^3\tilde{k}'+$$

$$+\,i\sum_{\substack{p_1+p_2=p\\ p_1>1,p_2>1}}\int\limits_0^1 d\tilde{s}_1\int\limits_0^1 d\tilde{s}_2\int\limits_{R^3}\langle\, g_{p_1}((\tilde{k}-\tilde{k}')\,\sqrt{\tilde{s}_1},\tilde{s}_1 s),\tilde{k}\,\rangle\,\cdot$$

$$P_{\tilde{k}}\, g_{p_2}(\tilde{k}'\sqrt{\tilde{s}_2},\tilde{s}_2 s)\,\exp\{-(1-\tilde{s}_1)\,|\tilde{k}-\tilde{k}'|^2-(1-\tilde{s}_2)|\tilde{k}'|^2\}\,d^3\tilde{k}'$$

$$+\,\frac{i}{s}\int\limits_0^1 d\tilde{s}_1\int\limits_{R^3}\langle\, g_{p-1}((\tilde{k}-\tilde{k}'))\,\sqrt{\tilde{s}_1}\,,\;\tilde{s}_1 s),\tilde{k}\rangle P_{\tilde{k}}\, v\left(\frac{\tilde{k}'}{\sqrt{s}},0\right)\cdot$$

$$\exp\{-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2\;-\;|\tilde{k}'|^2\}\,d^3\tilde{k}' \tag{6'}$$

The function $g_2(\tilde{k},s)$ has a singularity at $s=0$ even in the case of functions with compact support: for small s its values are of order $\frac{1}{\sqrt{s}}$. This singularity is integrable and all $g_p(k,s)$, $p>2$, are bounded. The singularity is connected with our choice of the coordinates $\tilde{k},\tilde{k}'$.

The formulas (4)-(6) or (4′)-(6′) resemble convolutions in probability theory. For example, if $C=\mathsf{supp}\,v(k,0)$ then $\mathsf{supp}\,h_p=\underbrace{C+C+\cdots+C}_{p\text{ times}}$. Therefore it is natural to expect that

h_p and g_p satisfy some form of the limit theorem of probability theory. This question will be discussed in more detail in the next sections.

Make another change of variables. Assume that we have some p. The terms in $(6')$ with $p_1 \le p^{1/2}$ and $p_2 \le p^{1/2}$ will be called boundary terms. They will be treated as remainder terms and will be estimated later. Suppose that we have some number $\tilde{k}^{(0)}$ which later will be assumed to be sufficiently large. Introduce the vector $\widetilde{\mathcal{K}}^{(r)} = (0, 0, r\tilde{k}^{(0)})$. These will be the points near which all g_r will be concentrated, $p^{1/2} \le r \le p - p^{1/2}$. We write $\tilde{k} = \widetilde{\mathcal{K}}^{(r)} + \sqrt{r} \cdot Y, Y \in R^3$. Thus instead of $\tilde{k}$ we have the new variable $Y = (Y_1, Y_2, Y_3)$ which typically will take values $O(1)$. Put $\tilde{\kappa}^{(0)} = (0, 0, \tilde{k}^{(0)})$.

In all integrals over $\tilde{s}_1, \tilde{s}_2$ in $(6')$ make another change of variables $1 - \tilde{s}_j = \frac{\theta_j}{p_j^2}$, $j = 1, 2$. Instead of the variable of integration $\tilde{k}'$ introduce Y' where $\tilde{k}' = \widetilde{\mathcal{K}}^{(p_2)} + \sqrt{p}Y'$. We write $\tilde{g}_r(Y, s) = g_r(\widetilde{\mathcal{K}}^{(r)} + \sqrt{r}\, Y, s)$, $\gamma = \frac{p_1}{p}$, $\frac{p_2}{p} = 1 - \gamma$. Then from $(6')$

$$\tilde{g}_p(Y, s) \;=\; g_p(\tilde{\mathcal{K}}^{(p)} \;+\; \sqrt{p}\, Y, s) \;=\; p^{5/2} \left[i \sum_{\substack{p_1, p_2 > \sqrt{p} \\ p_1 + p_2 = p}} \int_0^{p_1^2} d\theta_1 \int_0^{p_2^2} d\theta_2 \cdot \frac{1}{p_1^2 \cdot p_2^2} \cdot \right.$$

$$\int_{R^3} \left\langle \tilde{g}_{p_1} \left(\frac{Y - Y'}{\sqrt{\gamma}}, \left(1 - \frac{\theta_1}{p_1^2}\right) s \right), \tilde{\kappa}^{(0)} + \frac{Y}{\sqrt{p}} \right\rangle \cdot P_{\tilde{\kappa}^{(0)} + \frac{Y}{\sqrt{p}}}\, \tilde{g}_{p_2} \left(\frac{Y'}{\sqrt{1 - \gamma}}, \left(1 - \frac{\theta_2}{p_2^2}\right) s \right) \cdot$$

$$\left. \exp \left\{ -\theta_1 \left| \tilde{\kappa}^{(0)} + \frac{Y - Y'}{\sqrt{p} \cdot \gamma} \right|^2 - \theta_2 \left| \tilde{\kappa}^{(0)} + \frac{Y'}{\sqrt{p}\,(1 - \gamma)} \right|^2 \right\} d^3\, Y' \right]. \tag{7}$$

This is the main recurrent relation which we shall study in the next sections. It is of some importance that in front of (7) we have the factor $p^{5/2}$ and inside the sum the factor $\frac{1}{p_1^2} \cdot \frac{1}{p_2^2}$. Both are connected with the new scaling inherent to the Navier-Stokes system.

§3. The Renormalization Group Equation

As $p \longrightarrow \infty$ the recurrent equation (7) takes some limiting form which will be derived in this section. All remainders which appear in this way are listed and estimated in §8.

The main contribution to (7) comes from p_1, p_2 of order p. If $Y, Y' = O(1)$ then $\frac{Y - Y'}{\sqrt{p}}$, $\frac{Y'}{\sqrt{p}}$ are small compared to $\tilde{\kappa}^{(0)} = (0, 0, \tilde{k}^{(0)})$. Therefore the Gaussian term in (7) can be replaced

by $\exp\{-(\theta_1+\theta_2)|\tilde{k}^{(0)}|^2\}$, $\tilde{s}_1$ and $\tilde{s}_2$ can be replaced by 1 and the integrations over θ_1, θ_2 and Y' can be done separately. Thus instead of (7) we get a simpler recurrent relation:

$$\tilde{g}_p(Y,s) \,=\, \frac{i}{|\tilde{k}^{(0)}|^4}\, p^{5/2} \sum_{\substack{p_1,p_2>p^{1/2} \\ p_1+p_2=p}} \frac{1}{p_1^2\,\cdot\,p_2^2}\cdot \int\limits_{R^3} \left\langle \tilde{g}_{p_1}\left(\frac{Y-Y'}{\sqrt{\gamma}}\,,s\right),\tilde{\kappa}^{(0)}\,+\,\frac{Y}{\sqrt{p}}\right\rangle \cdot$$

$$\cdot\, P_{\tilde{\kappa}^{(0)}+\frac{Y}{\sqrt{p}}}\,\tilde{g}_{p_2}\left(\frac{Y'}{\sqrt{(1-\gamma)}},s\right)\,d^3\,Y'\,. \tag{8}$$

In view of incompressibility

$$\left\langle \tilde{g}_{p_1}\left(\tfrac{Y-Y'}{\sqrt{\gamma}},s\right),\;\tilde{\kappa}^{(0)}\,+\,\tfrac{Y}{\sqrt{p}}\right\rangle =$$

$$=\;\tfrac{1}{p_1}\left\langle g_{p_1}\left(\kappa^{(0)}\,\cdot\,p_1\,+\,\tfrac{Y-Y'}{\sqrt{\gamma}}\,\cdot\,\sqrt{p_1},s\right),\,\kappa^{(0)}p_1\,+\,Y\,\cdot\,\gamma\sqrt{p}\right\rangle$$

$$=\;\tfrac{1}{p_1}\,<\,g_{p_1}\left(\kappa^{(0)}p_1\,+\,\tfrac{Y-Y'}{\sqrt{\gamma}}\,\cdot\,\sqrt{p_1}\,,s\right),\,\kappa^{(0)}\,p_1\,+\,\tfrac{Y-Y'}{\sqrt{\gamma}}\,\sqrt{p_1}\,>\,+$$

$$+\;\tfrac{1}{p_1}\,<\,g_{p_1}\left(\kappa^{(0)}\,p_1\,+\,\tfrac{Y-Y'}{\sqrt{\gamma}}\,\cdot\,\sqrt{p_1},s\right),\,Y\gamma\,\cdot\,\sqrt{p}-(Y-Y')\,\sqrt{p}\;>=$$

$$=\;\tfrac{1}{\sqrt{p_1}}\cdot\,<\,\tilde{g}_{p_1}\left(\tfrac{Y-Y'}{\sqrt{\gamma}},s\right)\,,\;\tfrac{Y-Y'}{\sqrt{\gamma}}\;>\;\cdot(\gamma-1)\,+$$

$$+\;\tfrac{1}{\sqrt{p_2}}\,<\,\tilde{g}_{p_1}\left(\tfrac{Y-Y'}{\sqrt{\gamma}},s\right)\,,\,Y'\sqrt{(1-\gamma)}\;> \tag{9}$$

Write $\tilde{g}_p$ in the form

$$\tilde{g}_p(Y,s)\,=\,(G_1^{(p)}(Y,s)\,,\,G_2^{(p)}(Y,s)\,,\,\frac{1}{\sqrt{p}}\,F^{(p)}(Y,s))\,. \tag{10}$$

Since $\tilde{k}=\tilde{\kappa}^{(0)}\cdot p\,+\,Y\sqrt{p}$, the incompressibility implies

$$<\,g_r(\tilde{k},s)\,,\,\tilde{k}\;>=<\,g_r(\tilde{k},s)\,,\,\frac{\tilde{k}}{r}\;>=\,0 \tag{11}$$

and for $Y\,=\,O(1)$

$$\frac{Y_1}{\sqrt{r}}\,\cdot\,G_1^{(r)}(Y,s)\,+\,\frac{Y_2}{\sqrt{r}}\,G_2^{(r)}(Y,s)\,+\,\frac{\tilde{k}^{(r)}}{\sqrt{r}}\,\cdot\,F^{(r)}(Y,s)\,=\,O\left(\frac{1}{r}\right)\,. \tag{12}$$

In our approximation we replace (12) by

$$Y_1 G_1^{(r)}(Y,s) \,+\, Y_2 G_2^{(r)}(Y,s) \,+\, F^{(r)}(Y,s) \,=\, 0\,. \tag{13}$$

Thus for given Y_1, Y_2, Y_3 the component F_r can be expressed through $G_1^{(r)}, G_2^{(r)}$. This remains to be true even if we do not neglect the rhs of (13). Return back to (9). From (13)

$$< \ \tilde g_{p_1}\left(\tfrac{Y-Y'}{\sqrt{\gamma}}, s\right), \tilde\kappa^{(0)} \,+\, \tfrac{Y}{\sqrt{p}} >= \tfrac{1}{\sqrt{p}}\left[\tfrac{\gamma-1}{\sqrt{\gamma}} \ < \ \tilde g_{p_1}\left(\tfrac{Y-Y'}{\sqrt{\gamma}}, s\right), \ \tfrac{Y-Y'}{\sqrt{\gamma}} \ > \right.$$

$$+ \ \sqrt{1-\gamma} \ < \tilde g_{p_1}\left(\tfrac{Y-Y'}{\sqrt{\gamma}}, s\right) \ , \tfrac{Y'}{\sqrt{1-\gamma}} > \ ] \ =$$

$$= \ \tfrac{1}{\sqrt{p}}\left[\tfrac{\gamma-1}{\sqrt{\gamma}}\left(\tfrac{Y_1-Y_1'}{\sqrt{\gamma}}\, G_1^{(p_1)}\left(\tfrac{Y-Y'}{\sqrt{\gamma}}, s\right) \,+\, \tfrac{Y_2-Y_2'}{\sqrt{\gamma}}\, G_2^{(p_1)}\left(\tfrac{Y-Y'}{\sqrt{\gamma}},\ s\right)\right.\right.$$

$$+ \ \tfrac{Y_3-Y_3'}{\sqrt{\gamma}} \ \cdot \ \tfrac{1}{\sqrt{p_1}} \ \cdot \ F^{(p_1)}\left(\tfrac{Y-Y'}{\sqrt{\gamma}}, s\right) \,+\, \sqrt{(1-\gamma)}\left(\tfrac{Y_1'}{\sqrt{1-\gamma}}\, G_1^{(p_1)}\left(\tfrac{Y-Y'}{\sqrt{\gamma}}, s\right)\right.$$

$$\left.\left. + \ \tfrac{Y_2'}{\sqrt{1-\gamma}}\, G_2^{(p_1)}\left(\tfrac{Y-Y'}{\sqrt{\gamma}}, s\right) \,+\, \tfrac{1}{\sqrt{p_2}}\, \tfrac{Y_3'}{\sqrt{1-\gamma}}\, F^{(p_1)}\left(\tfrac{Y-Y'}{\sqrt{\gamma}}, s\right)\right)\right]\,. \tag{14}$$

In our approximation the inner product in (14) can be replaced by

$$\tfrac{1}{\sqrt{p}}\left[\tfrac{\gamma-1}{\sqrt{\gamma}}\left(\tfrac{Y_1-Y_1'}{\sqrt{\gamma}}\, G_1^{(p_1)}\left(\tfrac{Y-Y'}{\sqrt{\gamma}}, s\right) \,+\, \tfrac{Y_2-Y_2'}{\sqrt{\gamma}}\, G_2^{(p_1)}\left(\tfrac{Y-Y'}{\sqrt{\gamma}}, s\right)\right)\right.$$
$$\left. + \sqrt{1-\gamma}\left(\frac{Y_1'}{\sqrt{1-\gamma}}\, G_1^{(p_1)}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right) \,+\, \frac{Y_2}{\sqrt{1-\gamma}}\, G_2^{(p_1)}\left(\frac{Y-Y'}{\sqrt{\gamma}}\ s\right)\right)\right] \tag{15}$$

According to the definition of the projector

$$P_{\tilde\kappa^{(0)}+\frac{Y}{\sqrt{p}}}\, \tilde g_{p_2}\left(\tfrac{Y'}{\sqrt{1-\gamma}}, s\right) \,=\, \tilde g_{p_2}\left(\tfrac{Y'}{\sqrt{1-\gamma}}, s\right)$$

$$-\frac{< \tilde g_{p_2}\left(\frac{Y'}{\sqrt{1-\gamma}}, s\right), \ \tilde\kappa^{(0)} \,+\, \frac{Y}{\sqrt{p}} > \ \cdot (\tilde\kappa^{(0)} + \frac{Y}{\sqrt{p}})}{< \tilde\kappa^{(0)} \,+\, \frac{Y}{\sqrt{p}}, \tilde\kappa^{(0)} \,+\, \frac{Y}{\sqrt{p}} >} \,=\, \tilde g_{p_2}\left(\frac{Y'}{\sqrt{1-\gamma}}, s\right) \,+\, O\left(\frac{1}{\sqrt{p_2}}\right)\,. \tag{16}$$

This shows that in the main order of magnitude the projector is the identity operator and we come to a simpler recurrent relation instead of (8):

$$\tilde{g}_p(Y,s) \;=\; i\,\lambda_1 \,\cdot\, \sum_{\substack{p_1,p_2 \geq p^{1/2} \\ p_1+p_2=p}} \frac{p^2}{p_1^2\,\cdot\,p_2^2} \int_{R^3} \left[\frac{\gamma-1}{\sqrt{\gamma}} \left(\frac{Y_1-Y_1'}{\sqrt{\gamma}} \,\cdot \right.\right.$$

$$G_1^{(p_1)} \left(\frac{Y-Y'}{\sqrt{\gamma}}, s \right) \;+\; \frac{Y_2-Y_2'}{\sqrt{\gamma}}\, G_2^{(p_1)} \left. \left(\frac{Y-Y'}{\sqrt{\gamma}}, s \right) \right) \;+\; \sqrt{1-\gamma}$$

$$\left. \left(\frac{Y_1'}{\sqrt{1-\gamma}}\, G_1^{(p_1)} \left(\frac{Y-Y'}{\sqrt{\gamma}}, s \right) \;+\; \frac{Y_2'}{\sqrt{1-\gamma}}\, G_2^{(p_1)} \left(\frac{Y-Y'}{\sqrt{\gamma}}, s \right) \right) \right] \,\cdot\, \tilde{g}_{p_2} \left(\frac{Y'}{\sqrt{1-\gamma}}, s \right) \, d^3Y' \,. \tag{17}$$

The main assumption which we shall check below in the next sections concerns the asymptotic form of $\tilde{g}_p(Y,s)$ as $p \longrightarrow \infty$: for some interval $S^{(p)} \;=\; [S_-^{(p)},\; S_+^{(p)}]$ on the time axis and some Λ, positive $\sigma^{(1)}$, $\sigma^{(2)}$ and for all $r < p$

$$\tilde{g}_r(Y,s) \;=\; \Lambda^{r-1} r \,\cdot\, \frac{\sigma^{(1)}}{2\pi} \exp\left\{ -\frac{\sigma^{(1)}}{2} \left(|Y_1|^2 \,+\, |Y_2^2| \right) \right\} \,\cdot\, \sqrt{\frac{\sigma^{(2)}}{2\pi}} \exp\left\{ -\frac{\sigma^{(2)}}{2} |Y_3|^2 \right\} \,\cdot$$

$$\left(H_1(Y_1,Y_2,Y_3) \;+\; \delta_1^{(r)}(Y,s),\; H_2(Y_1,Y_2,Y_3) \;+\; \delta_2^{(r)}(Y,s)\,, \delta_3^{(r)}(Y,s) \right) \tag{18}$$

where $\delta_j^{(r)}(Y,s) \longrightarrow 0$ as $r \longrightarrow \infty$, $j = 1,2,3$. Later we shall explain in more detail in what sense the convergence to zero takes place. The substitution of (18) into (17) gives

$$\tilde{g}_p(Y,s) \;=\; \frac{i}{|\tilde{k}^{(p)}|^4} \,\cdot\, p \,\cdot\, \Lambda^{p-2} \,\cdot$$

$$\sum_{\gamma=\frac{p_1}{p}} \frac{1}{p} \cdot \gamma^{\frac{1}{2}} (1-\gamma)^{\frac{1}{2}} \,\cdot \int_{R^3} \left[\frac{\gamma-1}{\sqrt{\gamma}} \,\cdot\, \left(\frac{Y_1-Y_1'}{\sqrt{\gamma}} \,\cdot\, H_1 \left(\frac{Y-Y'}{\sqrt{\gamma}} \right) + \right.\right.$$

$$+ \left. \frac{Y_2-Y_2'}{\sqrt{\gamma}}\, H_2 \left(\frac{Y-Y'}{\sqrt{\gamma}} \right) \right) \;+\; \sqrt{1-\gamma} \left(\frac{Y_1'}{\sqrt{1-\gamma}}\, H_1 \left(\frac{Y-Y'}{\sqrt{\gamma}} \right) + \right.$$

$$+ \left.\left. \frac{Y_2'}{\sqrt{1-\gamma}}\, H_2 \left(\frac{Y-Y'}{\sqrt{\gamma}} \right) \right) \right] \, H \left(\frac{Y'}{\sqrt{1-\gamma}} \right) \,\cdot$$

$$\frac{\sigma^{(1)}}{2\pi\gamma} \,\cdot\, \exp\left\{ -\frac{\sigma^{(1)}}{2} \left(\frac{|Y_1-Y_1'|^2 \,+\, |Y_2-Y_2'|^2}{\gamma} \right) \right\} \,\cdot$$

$$\frac{\sigma^{(1)}}{2\pi(1-\gamma)} \exp\left\{-\frac{\sigma^{(1)}}{2}\frac{|Y_1'|^2 + |Y_2'|^2}{1-\gamma}\right\} \cdot$$

$$\sqrt{\frac{\sigma^{(2)}}{2\pi\gamma}} \exp\left\{-\frac{\sigma^{(2)}}{2}\frac{|Y_3 - Y_3'|^2}{\gamma}\right\} \cdot \sqrt{\frac{\sigma^{(2)}}{2\pi(1-\gamma)}} \exp\left\{-\frac{\sigma^{(2)}}{2}\frac{|Y_3'|^2}{1-\gamma}\right\} d^3Y' . \tag{19}$$

Here

$$H\left(\frac{Y'}{\sqrt{1-\gamma}}\right) = \left(H_1\left(\frac{Y_1'}{\sqrt{1-\gamma}}, \frac{Y_2'}{\sqrt{1-\gamma}}, \frac{Y_3'}{\sqrt{1-\gamma}}\right),\right.$$

$$\left. H_2\left(\frac{Y_1'}{\sqrt{1-\gamma}}, \frac{Y_2'}{\sqrt{1-\gamma}}, \frac{Y_3'}{\sqrt{1-\gamma}}\right), 0\right) .$$

We do not mention explicitly the dependence of H on s.

The last sum looks like a Riemannian integral sum whose limit takes the form as $p \longrightarrow \infty$:

$$\Lambda \exp\left\{-\tfrac{\sigma^{(1)}}{2}\left(|Y_1|^2 + |Y_2|^2\right)\right\} \cdot \frac{\sigma^{(1)}}{2\pi} \cdot \exp\left\{-\frac{\sigma^{(2)}|Y_3|^2}{2}\right\} \sqrt{\frac{\sigma^{(2)}}{2\pi}}\, H(Y)$$

$$= \quad \frac{i}{|\tilde{k}^{(0)}|^4} \int_0^1 \gamma^{\frac{1}{2}} (1-\gamma)^{\frac{1}{2}}\, d\gamma \int_{R^3} \frac{\sigma^{(1)}}{2\pi\gamma} \exp\left\{-\frac{\sigma^{(1)}(|Y_1 - Y_1'|^2 + |Y_2 - Y_2'|^2)}{2\gamma}\right\}$$

$$\frac{\sigma^{(1)}}{2\pi(1-\gamma)} \exp\left\{-\frac{\sigma^{(1)}(|Y_1'|^2 + |Y_2'|^2)}{2(1-\gamma)}\right\} \cdot \sqrt{\frac{\sigma^{(2)}}{2\pi\gamma}} \exp\left\{-\frac{\sigma^{(2)}|Y_3 - Y_3'|^2}{2\gamma}\right\} \cdot$$

$$\sqrt{\frac{\sigma^{(2)}}{2\pi(1-\gamma)}} \exp\left\{-\frac{\sigma^{(2)}|Y_3'|^2}{2(1-\gamma)}\right\} \left[-\frac{\gamma-1}{\sqrt{\gamma}}\left(\frac{Y_1 - Y_1'}{\sqrt{\gamma}} H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right) + \frac{Y_2 - Y_2'}{\sqrt{\gamma}} H_2\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right.$$

$$\left. + \gamma^{\frac{1}{2}}(1-\gamma)\left(\frac{Y_1'}{\sqrt{1-\gamma}} H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right) + \frac{Y_2'}{\sqrt{1-\gamma}} H_2\left(\frac{Y-Y'}{\sqrt{1-\gamma}}\right)\right)\right] \cdot H\left(\frac{Y'}{\sqrt{1-\gamma}}\right) d^3Y' . \tag{20}$$

The integral over Y_3 is the usual convolution. Therefore we can look for functions H_1, H_2 depending only on Y_1, Y_2, i.e. $H_1(Y) = H_1(Y_1, Y_2)$, $H_2(Y) = H_2(Y_1, Y_2)$. Write down the equation for H_1, H_2 which does not contain Y_3:

$$\exp\left\{-\tfrac{\sigma^{(1)}}{2}|Y|^2\right\}\cdot\frac{\sigma^{(1)}}{2\pi}\cdot H(Y) \;=\; \int_0^1 d\gamma \int\limits_{R^2} \frac{\sigma^{(1)}}{2\pi\gamma}\cdot\exp\left\{-\frac{\sigma^{(1)}|Y-Y'|^2}{2\gamma}\right\}\cdot\frac{\sigma^{(1)}}{2\pi(1-\gamma)}\cdot$$

$$\exp\left\{-\tfrac{\sigma^{(1)}}{2(1-\gamma)}\cdot|Y'|^2\right\}\left[-(1-\gamma)^{3/2}\left(\tfrac{Y_1-Y_1'}{\sqrt{\gamma}}\cdot H_1\left(\tfrac{Y-Y'}{\sqrt{\gamma}}\right)+\tfrac{Y_2-Y_2'}{\sqrt{\gamma}}H_2\left(\tfrac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right.$$

$$\left.+\gamma^{\frac{1}{2}}(1-\gamma)\left(\frac{Y_1'}{\sqrt{1-\gamma}}H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)+\frac{Y_2'}{\sqrt{1-\gamma}}H_2\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right]\cdot H\left(\frac{Y'}{\sqrt{1-\gamma}}\right)d^2Y'\,. \tag{21}$$

Here $Y = (Y_1, Y_2)$, $Y' = (Y_1', Y_2')$, $H(Y) = (H_1(Y_1, Y_2), H_2(Y_1, Y_2)$. This is our main equation for the fixed point of the renormalization group which we shall analyze in the next section (see also §7).

§4. Analysis of the Equation (21)

The solutions to the equation (21) have a natural scaling with respect to the parameter $\sigma = \sigma^{(1)}$. Namely, if we solve the equation (21) for $\sigma = 1$ and let the corresponding solution be $H(Y)$, then the general solution for arbitrary σ is given by the formula

$$H_\sigma(Y) \;=\; \sqrt{\sigma}\,H(\sqrt{\sigma}Y)\,. \tag{22}$$

This is analogous to the usual scaling of the Gaussian fixed point in probability theory. Thus, it is enough to consider the equation (21) for $\sigma = 1$. We shall show that there exists a three-parameter family of solutions to the equation (21) for $\sigma = 1$. The equation (21) takes a simpler form if we use expansions over Hermite polynomials. All necessary facts about Hermite polynomials are collected in the Appendix 1. For $H(Y_1, Y_2) = (H_1(Y_1, Y_2), H_2(Y_1, Y_2))$, we write

$$H_j(Y_1, Y_2) \;=\; \sum_{m_1, m_2 \geq 0} h^{(j)}(m_1, m_2)\,He_{m_1}(Y_1)\,He_{m_2}(Y_2), \quad j = 1, 2 \tag{23}$$

where $He_m(z)$ are the Hermite polynomials of degree m with respect to the Gaussian density $\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}z^2\right\}$. We have (see (42)):

$$zHe_m(z) \;=\; He_{m+1}(z) \;+\; mHe_{m-1}(z)\,,\; m > 0 \tag{24}$$

and

$$He_0(z) = 1, \qquad\qquad zHe_0(z) = z = He_1(z)\,.$$

Also we use the formula (see (43))

$$\int\limits_{\mathbb{R}^1} He_{m_1}\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{|Y-Y'|^2}{2\gamma}\right\} He_{m_2}\left(\frac{Y'}{\sqrt{1-\gamma}}\right)\frac{1}{\sqrt{2\pi}}$$

$$\exp\left\{-\frac{|Y'|^2}{2(1-\gamma)}\right\}dY' = \gamma^{\frac{m_1+1}{2}}(1-\gamma)^{\frac{m_2+1}{2}} He_{m_1+m_2}(Y)\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{|Y|^2}{2}\right\}. \qquad (25)$$

Substituting (23) into (21) and using (24), (25), we come to the system of equations for the coefficients $h(m_1, m_2)$ which is equivalent to (21):

$$h^{(j)}(m_1,m_2) = \sum_{\substack{m_1'+m_1''=m_1\\ m_2'+m_2''=m_2}} J^{(1)}_{m'm''}\cdot\left\{(B_1h^{(1)})(m_1',m_2') + (B_2h^{(2)})(m_1',m_2')\right\}h^{(j)}(m_1'',m_2'')$$

$$+J^{(2)}_{m'm''}\cdot\left\{h^{(1)}(m_1',m_2')\,(B_1h^{(j)})(m_1'',m_2'') + h^{(2)}(m_1',m_2')\,(B_2h^{(j)})(m_1'',m_2'')\right\} \qquad (26)$$

where $m' = m_1' + m_2'$, $m'' = m_1'' + m_2''$ and

$$\begin{cases} J^{(1)}_{m'm''} = -\displaystyle\int_0^1 \gamma^{\frac{m'}{2}}(1-\gamma)^{\frac{m''+3}{2}}d\gamma \\ J^{(2)}_{m'm''} = \displaystyle\int_0^1 \gamma^{\frac{m'+1}{2}}(1-\gamma)^{\frac{m''+2}{2}}d\gamma \end{cases} \qquad (27)$$

$$(B_1\,h^{(j)})\,(m_1',m_2') = h^{(j)}\,(m_1'-1,m_2') + (m_1'+1)\,h^{(j)}(m_1'+1,m_2')$$

$$(B_2\,h^{(j)})\,(m_1',m_2') = h^{(j)}\,(m_1',m_2'-1) + (m_2'+1)\,h^{(j)}\,(m_1',m_2'+1)$$

To simplify the system (26), we shall look for solutions with $h^{(j)}(0,0) = 0$, $j = 1,2$. Below we sometimes write $h^{(j)}(m_1,m_2)$ as $h^{(j)}_{m_1m_2}$ or $h^{(j)}_{m_1,m_2}$ when there is no confusion. Similar conventions will be applied to $J^{(j)}(m_1,m_2)$. For $m_1+m_2 = 1$, we have

$$
\begin{cases}
h_{10}^{(1)} & = & J_{01}^{(1)} \cdot (h_{10}^{(1)} + h_{01}^{(2)}) \cdot h_{10}^{(1)} + J_{10}^{(2)} \cdot (h_{10}^{(1)} h_{10}^{(1)} + h_{10}^{(2)} h_{01}^{(1)}) \\
h_{01}^{(1)} & = & J_{01}^{(1)} \cdot (h_{10}^{(1)} + h_{01}^{(2)}) \cdot h_{01}^{(1)} + J_{10}^{(2)} \cdot (h_{01}^{(1)} h_{10}^{(1)} + h_{01}^{(2)} h_{01}^{(1)}) \\
h_{10}^{(2)} & = & J_{01}^{(1)} \cdot (h_{10}^{(1)} + h_{01}^{(2)}) \cdot h_{10}^{(2)} + J_{10}^{(2)} \cdot (h_{10}^{(1)} h_{10}^{(2)} + h_{10}^{(2)} h_{01}^{(2)}) \\
h_{01}^{(2)} & = & J_{01}^{(1)} \cdot (h_{10}^{(1)} + h_{01}^{(2)}) \cdot h_{01}^{(2)} + J_{10}^{(2)} \cdot (h_{01}^{(1)} h_{10}^{(2)} + h_{01}^{(2)} h_{01}^{(2)})
\end{cases}
$$

where $J_{01}^{(1)} = -1/3$ and $J_{10}^{(2)} = 1/6$. There are two cases:

<u>Case 1.</u> $h_{10}^{(1)} + h_{01}^{(2)} = -6$. In this case $(h_{10}^{(1)}, h_{01}^{(1)}, h_{10}^{(2)}, h_{01}^{(2)})$ only needs to satisfy:

$$(h_{10}^{(1)} + 3)^2 = 9 - h_{01}^{(1)} h_{10}^{(2)}$$

This is a two parameter family of solutions.

<u>Case 2.</u> $h_{10}^{(1)} + h_{01}^{(2)} \neq -6$. In this case $(h_{10}^{(1)}, h_{01}^{(1)}, h_{10}^{(2)}, h_{01}^{(2)})$ can be uniquely determined and we have $h_{10}^{(1)} = h_{01}^{(2)} = -2$, $h_{01}^{(1)} = h_{10}^{(2)} = 0$.

For the rest of this paper we shall consider only the case 2 for which $h_{10}^{(1)} = h_{01}^{(2)} = -2$, $h_{01}^{(1)} = h_{10}^{(2)} = 0$. Let us write down the recurrent relations for $m_1 + m_2 = 2$, $j = 1, 2$:

$$
\begin{cases}
h_{20}^{(j)} & = & -(2J_{20}^{(2)} + 4J_{02}^{(1)} + 4J_{11}^{(2)}) h_{20}^{(j)} + 2J_{11}^{(1)} \cdot h_{10}^{(j)} \cdot h_{20}^{(1)} + h_{10}^{(j)} \cdot J_{11}^{(1)} \cdot h_{11}^{(2)} \\
h_{11}^{(j)} & = & -(2J_{20}^{(2)} + 4J_{02}^{(1)} + 4J_{11}^{(2)}) h_{11}^{(j)} + J_{11}^{(1)} \dot{h}_{01}^{(j)} \cdot (2h_{20}^{(1)} + h_{11}^{(2)}) + J_{11}^{(1)} \cdot h_{10}^{(j)} \cdot (h_{11}^{(1)} + 2h_{02}^{(2)}) \\
h_{02}^{(j)} & = & -(2J_{20}^{(2)} + 4J_{02}^{(1)} + 4J_{11}^{(2)}) h_{02}^{(j)} + 2J_{11}^{(1)} \cdot h_{01}^{(j)} \cdot h_{02}^{(2)} + h_{01}^{(j)} \cdot J_{11}^{(1)} \cdot h_{11}^{(1)}
\end{cases}
$$

It is not difficult to check that the only solution to the above system is $h_{20}^{(j)} = h_{02}^{(j)} = h_{11}^{(j)} = 0$. Solving the recurrent relations for $m_1 + m_2 = 3$ gives us:

$$
\begin{cases}
h_{03}^{(1)} = h_{30}^{(2)} = 0 \\
h_{12}^{(1)} = h_{03}^{(2)} \\
h_{21}^{(1)} = h_{12}^{(2)} \\
h_{30}^{(1)} = h_{21}^{(2)}
\end{cases}
$$

This shows that $(h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)})$ can be considered as free parameters. For any $p \geq 4$, the recurrent relations for $m_1 + m_2 = p$ form a linear system of equations for the variables

$\{h^{(j)}_{m_1,p-m_1}\}^{p}_{m_1=0}$ with coefficients depending on $h^{(j)}_{01}$ and $h^{(j)}_{10}$ only. In principle, they can be solved and an explicit expression for the solutions can be found. We emphasize here that if the free parameters take real values then the whole solution is also real.

It is not difficult to check that for any values of $(h^{(1)}_{12}, h^{(1)}_{21}, h^{(1)}_{30})$, one can find all $h^{(j)}_{m_1,m_2}$ $(m_1+m_2 \geq 4)$ by using (26). The solution we obtain is formal in the sense that it satisfies (26) but h_{m_1,m_2} with $m_1+m_2 = p$ may not decay as $p \longrightarrow \infty$. We are now ready to formulate the theorem concerning the existence of formal solutions to (26).

Theorem 4.1. *For any values of $(h^{(1)}_{12}, h^{(1)}_{21}, h^{(1)}_{30})$, there exists a unique formal solution to the recurrent equation (26).*

Thus, theorem 4.1 claims the existence of a three-parameter family of solutions of (21) parameterized by $h^{(1)}_{12}$, $h^{(1)}_{21}$ and $h^{(1)}_{30}$. It turns out that if $h^{(1)}_{12}$, $h^{(1)}_{21}$ and $h^{(1)}_{30}$ are sufficiently small, then $h^{(j)}_{m_1,m_2}$ decay as $m_1+m_2 = p$ tends to infinity. Let us say that $h^{(j)}_{m_1,m_2}$ has degree d if $m_1+m_2 = d$. For each $d \geq 4$, introduce the vector $h^{(d)} = (h^{(1)}_{0,d}, h^{(1)}_{1,d-1}, \ldots, h^{(1)}_{d,0}, h^{(2)}_{0,d}, \ldots, h^{(2)}_{d,0})^T$. The vector $h^{(d)}$ contains all terms of degree d. By the recurrent relation (26)

$$C^{(d)}\, h^{(d)} \;=\; b^{(d)} \tag{28}$$

where the vector $b^{(d)}$ contains terms of degree $\leq d-1$. Also $C^{(d)} \in \mathbb{R}^{(2d+2)\times(2d+2)}$ is a matrix:

$$C^{(d)}_{k\ell} \;=\; \begin{cases} 1 - \dfrac{16d-16+32k}{(d+1)(d+3)(d+5)}, & \text{if } 1 \leq k \;=\; l \leq d+1 \\[2ex] 1 - \dfrac{80d+80-32k}{(d+1)(d+3)(d+5)}, & \text{if } d+2 \leq k \;=\; l \leq 2d+2 \\[2ex] -\dfrac{32(d-k+2)}{(d+1)(d+3)(d+5)}, & \text{if } 2 \leq k \leq d+1, l = d+k \\[2ex] -\dfrac{32(k-d-1)}{(d+1)(d+3)(d+5)}, & \text{if } d+2 \leq k \leq 2d+1, l = k-d \\[2ex] 0, & \text{all other cases} \end{cases}$$

It is easy to check that if $d \geq 4$, then $C^{(d)}$ is nonsingular and as $d \longrightarrow \infty$, $C^{(d)}$ converges to the identity matrix. This observation immediately implies the following lemma:

Lemma 4.2. *Let $(C^{(d)})^{-1}$ be the inverse matrix of $C^{(d)}$ for $d \geq 4$. There exists an absolute constant $C_1 > 0$ such that for all $d \geq 4$*

$$\| \, (C^{(d)})^{-1} \, \| \leq \, C_1 \, .$$

We are now ready to derive an estimate which gives the decay of solutions of the recurrent relation (26).

Theorem 4.3. *If $|h_{12}^{(1)}| \leq \delta$, $|h_{21}^{(1)}| \leq \delta$, $|h_{30}^{(1)}| \leq \delta$ and δ is sufficiently small, then for some $C_2 > 0$, $0 < \rho < \frac{1}{4}$, we have*

$$\left| h_{m_1,m_2}^{(j)} \right| \leq C_2 \, \frac{\rho^{m_1+m_2}}{\Gamma\left(\frac{m_1+m_2+7}{2}\right)} \quad \forall \, m_1 \geq 0, \, m_2 \, \geq \, 0, \; j = 1, 2.$$

Proof. We begin by noting that $h_{m_1 m_2}^{(j)} = \, 0$ if $m_1 + m_2$ is even. This can be easily proven by using the recurrent relation (26) and the fact that $h_{00}^{(j)} = 0$ and $h_{m_1,m_2}^{(j)} = 0$ for $m_1 + m_2 = 2$. Let $0 < \rho_1 < 1$, ρ_1 will be chosen sufficiently small. We shall use induction on $m_1 + m_2$ where $m_1 + m_2$ is odd. According to the induction hypothesis

$$|h_{m_1,m_2}^{(j)}| \, \leq \, \frac{\rho_1^{m_1+m_2+2}}{\Gamma\left(\frac{m_1+m_2+7}{2}\right)} \, g(m_1 + m_2) \tag{29}$$

for every $3 \leq m_1 + m_2 \leq d - 2$ where $d \geq L$ is an odd number and L will be chosen later to be sufficiently large. Also g is a function to be specified later. We shall comment on the choice of L and verify the induction hypothesis for $3 \leq m_1 + m_2 \leq L$ later. Let us show that the same inequality holds for $m_1 + m_2 = d$. Without any loss of generality, let us consider $j = 1$. The case $j = 2$ is similar. Fix m_1 and let $b_{m_1}^{(d)}$ be the $(m_1 + 1)^{\text{th}}$ component of the vector $b^{(d)}$ in the equation (28). We now estimate $b_{m_1}^{(d)}$ using the induction hypothesis (29)

and the equation (26):

$$
\begin{aligned}
\left|b^{(d)}_{m_1}\right| \le & \sum_{m'=2}^{d-3} \left|J^{(1)}_{m',m''}\right| \cdot 2 \cdot \frac{\rho_1^{m'+3}}{\Gamma\left(\frac{m'+8}{2}\right)} \cdot \frac{\rho_1^{m''+2}}{\Gamma\left(\frac{m''+7}{2}\right)} \cdot (m'+1) g(m'+1) g(m'') \\
& + \sum_{m'=4}^{d-3} \left|J^{(1)}_{m',m''}\right| \cdot 2 \cdot \frac{\rho_1^{m'+1}}{\Gamma\left(\frac{m'+6}{2}\right)} \cdot \frac{\rho_1^{m''+2}}{\Gamma\left(\frac{m''+7}{2}\right)} \cdot (m'+1) \cdot g(m'-1) g(m'') \\
& + \sum_{m'=3}^{d-2} \left|J^{(2)}_{m',m''}\right| \cdot 2 \cdot \frac{\rho_1^{m'+2}}{\Gamma\left(\frac{m'+7}{2}\right)} \cdot \frac{\rho_1^{m''+3}}{\Gamma\left(\frac{m''+8}{2}\right)} \cdot (m'+1) \cdot (m''+1) \cdot g(m') g(m''+1) \\
& + \sum_{m'=3}^{d-4} \left|J^{(2)}_{m',m''}\right| \cdot 2 \cdot \frac{\rho_1^{m'+2}}{\Gamma\left(\frac{m'+7}{2}\right)} \cdot \frac{\rho_1^{m''+1}}{\Gamma\left(\frac{m''+6}{2}\right)} \cdot (m'+1) \cdot g(m') g(m''-1) \\
& + 12 \left(\left|J^{(1)}_{2,d-2}\right| + \left|J^{(1)}_{d-1,1}\right| + \left|J^{(2)}_{d-2,2}\right| + \left|J^{(2)}_{1,d-1}\right| \right) \cdot \frac{\rho_1^{d}}{\Gamma\left(\frac{d+5}{2}\right)} \cdot g(d-2)
\end{aligned}
$$

The last term in the rhs of the above inequality comes from the case where $h_{m_1'm_2'}$ or $h_{m_1''m_2''}$ is of degree one since the induction hypothesis holds only for $3 \le m_1+m_2 \le d-2$. Also in the estimation of the first four terms we use the fact that for fixed (m', m_1), there are at most $\min\{m'+1, m''+1\}$ tuples of $(m', m_1^{''}, m_2', m_2^{''})$ such that $m_1' + m_1^{''} = m_1$, $m_2' + m_2^{''} = m_2$, $m_1' + m_2' = m'$ and $m_1^{''} + m_2^{''} = m^{''}$. By (27), we have

$$
\left|J^{(1)}_{m',m''}\right| = \frac{\Gamma\left(\frac{m'+2}{2}\right)\Gamma\left(\frac{m''+5}{2}\right)}{\Gamma\left(\frac{m'+m''+7}{2}\right)}
$$

$$
\left|J^{(2)}_{m',m''}\right| = \frac{\Gamma\left(\frac{m'+3}{2}\right)\Gamma\left(\frac{m''+4}{2}\right)}{\Gamma\left(\frac{m'+m''+7}{2}\right)}
$$

and for some constant $C_3 > 0$

$$
\left|J^{(1)}_{2,d-2}\right| + \left|J^{(1)}_{d-1,1}\right| + \left|J^{(2)}_{d-2,2}\right| + \left|J^{(2)}_{1,d-1}\right| \le \frac{C_3}{d^2}
$$

Therefore

$$
\begin{aligned}
\left|b_{m_1}^{(d)}\right| \leq & \frac{2\rho_1^{d+5}}{\Gamma\left(\frac{d+7}{2}\right)} \cdot \sum_{m'=2}^{d-3} \frac{\Gamma\left(\frac{m'+2}{2}\right)\cdot(m'+1)}{\Gamma\left(\frac{m'+8}{2}\right)} \cdot \frac{\Gamma\left(\frac{m''+5}{2}\right)}{\Gamma\left(\frac{m''+7}{2}\right)} \cdot g(m'+1)g(m'') \\
& + \frac{2\rho_1^{d+3}}{\Gamma\left(\frac{d+7}{2}\right)} \cdot \sum_{m'=4}^{d-3} \frac{\Gamma\left(\frac{m'+2}{2}\right)\cdot(m'+1)}{\Gamma\left(\frac{m'+6}{2}\right)} \cdot \frac{\Gamma\left(\frac{m''+5}{2}\right)}{\Gamma\left(\frac{m''+7}{2}\right)} \cdot g(m'-1)g(m'') \\
& + \frac{2\rho_1^{d+5}}{\Gamma\left(\frac{d+7}{2}\right)} \cdot \sum_{m'=3}^{d-2} \frac{\Gamma\left(\frac{m'+3}{2}\right)\cdot(m'+1)}{\Gamma\left(\frac{m'+7}{2}\right)} \cdot \frac{\Gamma\left(\frac{m''+4}{2}\right)\cdot(m''+1)}{\Gamma\left(\frac{m''+8}{2}\right)} \cdot g(m')g(m''+1) \\
& + \frac{2\rho_1^{d+3}}{\Gamma\left(\frac{d+7}{2}\right)} \cdot \sum_{m'=3}^{d-4} \frac{\Gamma\left(\frac{m'+3}{2}\right)\cdot(m'+1)}{\Gamma\left(\frac{m'+7}{2}\right)} \cdot \frac{\Gamma\left(\frac{m''+4}{2}\right)}{\Gamma\left(\frac{m''+6}{2}\right)} \cdot g(m')g(m''-1) \\
& + \frac{\rho_1^{d+2}}{\Gamma\left(\frac{d+7}{2}\right)} \cdot \frac{C_3}{d^2} \cdot \frac{\Gamma\left(\frac{d+7}{2}\right)}{\Gamma\left(\frac{d+5}{2}\right)} \cdot \frac{12}{\rho_1^2} \cdot g(d-2) \\
\leq & \frac{\rho_1^{d+2}}{\Gamma\left(\frac{d+7}{2}\right)} \cdot \rho_1 \cdot C_4 \cdot \left(\sum_{m'=2}^{d-3} g(m'+1)g(m'') + \sum_{m'=4}^{d-3} g(m'-1)g(m'')\right. \\
& \left.\sum_{m'=3}^{d-2} g(m')g(m''+1) + \sum_{m'=3}^{d-4} g(m')g(m''-1)\right) + \frac{\rho_1^{d+2}}{\Gamma\left(\frac{d+7}{2}\right)} \cdot C_5 \cdot \frac{g(d-2)}{d\cdot\rho_1}
\end{aligned}
$$

where C_4, C_5 are some constants. Now we specify the choice of the function g. Let $g(m)$ be such that $g_1 = \alpha$ and

$$
g(m) = \sum_{p=1}^{m-1} g(p)g(m-p) \qquad \text{for } m > 1
$$

By the method of formal power series it is not difficult to show that

$$
g(m) = \frac{1}{2} \cdot \frac{(2m-1)!!}{m!} \cdot (2\alpha)^m
$$

Clearly, we have $const \leq \frac{g(m+1)}{g(m)} \leq const$, and this immediately gives us

$$
\begin{aligned}
\left|b_{m_1}^{(d)}\right| \leq & \frac{\rho_1^{d+2}}{\Gamma\left(\frac{d+7}{2}\right)} \cdot C_6 \cdot \sum_{m'=1}^{d} g(m')g(d-m') + \frac{\rho_1^{d+2}}{\Gamma\left(\frac{d+7}{2}\right)} \cdot \frac{C_6}{d\cdot\rho_1} g(d) \\
\leq & \frac{\rho_1^{d+2}}{\Gamma\left(\frac{d+7}{2}\right)} \cdot g(d) \cdot \left(C_6\rho_1 + \frac{C_6}{d\cdot\rho_1}\right)
\end{aligned}
$$

where $C_6 > 0$ is some constant. Now by Lemma 4.2, we obtain that

$$
|h_{m_1 m_2}| \leq \frac{\rho_1^{d+2}}{\Gamma\left(\frac{d+7}{2}\right)} g(d) \cdot C_1 \cdot \left(C_6\rho_1 + \frac{C_6}{d\cdot\rho_1}\right)
$$

Choose ρ_1 so small that $C_1C_6\rho_1 < \frac{1}{2}$ and $\rho_1 \cdot 4\alpha < \frac{1}{4}$. Then take L so large that $\frac{C_1C_6}{\rho_1 L} < \frac{1}{2}$. This clearly implies

$$|h_{m_1,m_2}| \le \frac{\rho_1^{d+2}}{\Gamma\left(\frac{d+7}{2}\right)} g(d)$$

We now justify the induction hypothesis (29). Recall that our free parameters are $h_{12}^{(1)}$, $h_{21}^{(1)}$ and $h_{30}^{(1)}$. It is easy to check that if we set $h_{12}^{(1)} = h_{21}^{(1)} = h_{30}^{(1)} = 0$, then $h_{m_1m_2} = 0$ for any $m_1 + m_2 \ge 2$. Since L is fixed, and $0 < |h_{12}^{(1)}| < \delta$, $0 < |h_{21}^{(1)}| < \delta$, $0 < |h_{30}^{(1)}| < \delta$ with sufficiently small δ, then the induction hypothesis is satisfied. A simple estimate on g gives that

$$g(m) \le (4\alpha)^m$$

Thus the theorem is proven if one takes $\rho = 4\alpha\rho_1$.

As it is stated our solutions of (20) are determined by five parameters $\sigma^{(1)}, h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)}, \sigma^{(2)}$. However, it turns out that these parameters are not independent and σ_1 can be expressed through ($h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)}$). Namely, let $G^{\sigma^{(1)}, h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)}, \sigma^{(2)}}\ (Y)$ be the solution of (20). Then

$$G^{(\sigma^{(1)}, h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)}, \sigma^{(2)})}\ (Y) \;=\; G^{(1, \sigma^{(1)}(h_{12}^{(1)}-1)+1\,,\ \sigma^{(1)}h_{21}^{(1)},\ \sigma^{(1)}(h_{30}^{(1)}-1)+1,\ \sigma^{(2)})}(Y)\,.$$

This equality is proved at the end of §6. We formulate now the final result concerning the existence of solutions of (21).

Theorem 4.2. *Let $\sigma^{(1)} > 0$, $\sigma^{(2)} > 0$ and $h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)}$ be sufficiently small. Then there exists a solution of (20) which has the following form*

$$G^{(\sigma^{(1)}, h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)}, \sigma^{(2)})}\ (Y_1, Y_2, Y_3) \;=\; \exp\left\{-\frac{\sigma^{(1)}}{2}\ \left(|Y_1|^2 \;+\; |Y_2|^2\right)\right\} \cdot$$

$$\cdot \frac{\sigma^{(1)}}{2\pi} \cdot \exp\left\{-\frac{\sigma^{(2)}}{2}|Y_3|^2\right\} \sqrt{\frac{\sigma^{(2)}}{2\pi}} \cdot \sqrt{\sigma^{(1)}}\, H^{(h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)})}\, (\sqrt{\sigma^{(1)}}\, Y_1, \sqrt{\sigma^{(1)}}\, Y_2)\,.$$

Here $H^{(h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)})}$ is the solution of (21) with the given $h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)}$.

As it was already mentioned the parameters σ_1, $h_{12}^{(1)}$, $h_{21}^{(1)}$, $h_{30}^{(1)}$, σ_2 are not independent and actually the set of solutions depends on four independent parameters (see Lemma 6.2).

From the estimate in Theorem 4.3 and from known asymptotic formulas for the Hermite polynomials it follows that the series giving $H^{(h_{12}^{(1)}, h_{21}^{(1)}, h_{30}^{(1)})}$ converges for every $Y = (Y_1, Y_2)$. Better estimates are also easily available.

§5. The Linearization Near Fixed Point

Denote $h_{12}^{(1)} = x^{(1)}$, $h_{21}^{(1)} = x^{(2)}$, $h_{30}^{(1)} = x^{(3)}$. Our fixed points have the following form

$$\begin{aligned} G^{(\sigma^{(1)},x^{(1)},x^{(2)},x^{(3)},\sigma^{(2)})} &= \frac{\sigma^{(1)}}{2\pi}\exp\left\{-\frac{\sigma^{(1)}(Y_1^2+Y_2^2)}{2}\right\}\cdot \\ &\cdot \sqrt{\frac{\sigma^{(2)}}{2\pi}}\exp\left\{-\frac{\sigma^{(2)}Y_3^2}{2}\right\}\left(H_1^{(\sigma^{(1)},\,x^{(1)},\,x^{(2)},\,x^{(3)})}(Y_1,Y_2),\ H_2^{(\sigma^{(1)},\,x^{(1)},\,x^{(2)},\,x^{(3)})}(Y_1,Y_2),\ 0\right) \end{aligned} \tag{30}$$

Recall that $H^{(\sigma^{(1)},\,x^{(1)},\,x^{(2)},\,x^{(3)})} = \sqrt{\sigma^{(1)}}H^{(1,\,x^{(1)},\,x^{(2)},\,x^{(3)})}(\sqrt{\sigma^{(1)}}Y_1,\sqrt{\sigma^{(1)}}Y_2)$ and $H^{(1,\,x^{(1)},\,x^{(2)},\,x^{(3)})}$ are described in §4.

The strategy of the proof of the main result is based on the method of renormalization group. At the p-th step of our procedure, we consider an interval on the time axis $S^{(p)} = \left[S_-^{(p)}, S_+^{(p)}\right]$ such that $S^{(p+1)} \subseteq S^{(p)}$. From our estimates it will follow that $\bigcap_p S^{(p)} = [S_-, S_+]$ is an interval of positive length. We want to find conditions under which $\tilde{g}_r(Y,s)$, $s \in S^{(p)}$, have a representation

$$\begin{gathered} \tilde{g}_r(Y,s) \;=\; \Lambda^{r-1}r\frac{\sigma^{(1)}}{2\pi}\exp\left\{-\frac{\sigma^{(1)}(Y_1^2+Y_2^2)}{2}\right\}\sqrt{\frac{\sigma^{(2)}}{2\pi}}\exp\left\{-\frac{\sigma^{(2)}Y_3^2}{2}\right\}\cdot \\ \left(H_1^{(\sigma^{(1)},x^{(1)},x^{(2)},x^{(3)})}(Y)+\delta_1^{(r)}(Y,s),\ H_2^{(\sigma^{(1)},x^{(1)},x^{(2)},x^{(3)})}(Y)+\delta_2^{(r)}(Y,s),\ \delta_3^{(r)}(Y,s)\right) \end{gathered}$$

where $\delta_1^{(r)}$, $\delta_2^{(r)}$, $\delta_3^{(r)}$ tend to zero as $r \to \infty$. The renormalization is based on the crucial observation (see above) that for large p, the sum over p_1 is a Riemannian integral sum for an integral over γ changing from 0 to 1. Let us write

$$\begin{aligned} \tilde{g}_r(Y,s)\,\Lambda^{-r+1}(r^{-1}\exp\left\{\frac{\sigma^{(1)}(Y_1^2+Y_2^2)}{2}+\frac{\sigma^{(2)}Y_3^2}{2}\right\}\left(\frac{2\pi}{\sigma^{(1)}}\right)\left(\frac{2\pi}{\sigma^{(2)}}\right)^{\frac{1}{2}} &= \\ = H^{(\sigma^{(1)},x^{(1)},x^{(2)},x^{(3)})}\,(Y_1,Y_2)\;+\;\delta^{(r)}(\gamma,Y,s)& \end{aligned} \tag{31}$$

where $\delta^{(r)}(\gamma,Y,s) = \left\{\delta_j^{(r)}(\gamma,Y,s),\ 1 \le j \le 3\right\} = \delta^{(p)}(\gamma,Y,s)$, $\gamma = \dfrac{r}{p}$. It is natural to consider the set of functions $\{\delta^{(p)}(\gamma,Y,s)\}$ as a small perturbation of our fixed point (30). Recall that the third component of $H^{(\sigma^{(1)},x^{(1)},x^{(2)},x^{(3)})}$ is zero because of incompressibility and $\delta_3^{(p)}$ can be found from the incompressibility condition. Clearly,

$$\delta^{(p+1)}(\gamma,Y,s) \;=\; \delta^{(p)}\left(\frac{p+1}{p}\gamma,Y,s\right),\ \ \gamma \le \frac{p}{p+1}\,.$$

The formula for $\delta^{(p+1)}(1,Y,s)$ follows from (21):

$$\exp\left\{-\frac{\sigma^{(1)}}{2}\left(|Y_1|^2\ +\ |Y_2|^2\right)\ -\ \frac{\sigma^{(2)}}{2}|Y_3|^2\right\}\cdot\ \frac{\sigma^{(1)}}{2\pi}\sqrt{\frac{\sigma^{(2)}}{2\pi}}\ \cdot\ \delta_j^{(p+1)}(1,Y,s)$$

$$=\int\limits_0^1 d\gamma\int\limits_{\mathbb{R}^3}\frac{\sigma^{(1)}}{2\pi\gamma}\ \cdot\ \sqrt{\frac{\sigma^{(1)}}{2\pi\gamma}}\cdot\ \frac{\sigma^{(2)}}{2\pi(1-\gamma)}\ \cdot\ \sqrt{\frac{\sigma^{(2)}}{2\pi(1-\gamma)}}\cdot$$

$$\exp\left\{-\frac{\sigma^{(1)}(|Y_1-Y_1'|^2+|Y_2-Y_2'|^2)}{2\pi\gamma}-\frac{\sigma^{(2)}|Y_3-Y_3'|^2}{2\pi\gamma}\ -\ \frac{\sigma^{(1)}(|Y_1'|^2+|Y_2'|^2)}{2\pi(1-\gamma)}-\right.$$

$$\left.-\ \frac{\sigma^{(2)}|Y_3'|^2}{2\pi(1-\gamma)}\right\}\ \left\{\left[-(1-\gamma)^{\frac{3}{2}}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}\ H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\ +\ \frac{Y_2-Y_2'}{\sqrt{\gamma}}\,H_2\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right.\right.$$

$$\left.+\,\gamma^{\frac{1}{2}}(1-\gamma)\ \left(\frac{Y_1'}{\sqrt{1-\gamma}}\,H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\ +\ \frac{Y_2'}{\sqrt{1-\gamma}}\,H_2\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\right]\ \delta_j^{(p+1)}\left(1-\gamma,\ \frac{Y'}{\sqrt{1-\gamma}},s\right)$$

$$+\ \left[-\left(1-\gamma\right)^{\frac{3}{2}}\ \left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}\,\delta_1^{(p+1)}\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}}\,,s\right)\ +\ \frac{Y_2-Y_2'}{\sqrt{\gamma}}\,\delta_2^{(p+1)}\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}}\,,s\right)\right)\right.$$

$$\left.+\,\gamma^{\frac{1}{2}}\,(1-\gamma)\ \left(\frac{Y_1'}{\sqrt{1-\gamma}}\,\delta_1^{(p+1)}\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}}\,,s\right)\ +\ \frac{Y_2'}{\sqrt{1-\gamma}}\,\delta_2^{(p+1)}\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}},s\right)\right)\right)\right]$$

$$\left.H_j\left(\frac{Y'}{\sqrt{1-\gamma}},s\right)\right\}\ d^3Y',\qquad\qquad j\ =\ 1,2 \tag{32}$$

We did not include in the last expression terms which are quadratic in δ because in this section we consider only the linearized map.

Another way to introduce the semi-group of linearized maps is the following. Take $\theta>0$ which later will tend to zero. Denote $\gamma_j=(1+\theta)^{-j}$, $j=0,1,2,\dots$ Our semigroup will act on the space Δ of functions $\delta(\gamma,Y)$ with values in C^3 such that

1. for each γ, $0\le\gamma\le 1$, the function $\delta(\gamma,Y)$ belongs to the Hilbert space $L^2=L^2\left(R^3\right)$ of square-integrable functions with respect to the weight $\left(\frac{\sigma^{(1)}}{2\pi}\right)^{\frac{3}{2}}\exp\{-\frac{\sigma^{(1)}Y^2}{2}\}$, $Y=(Y_1,Y_2,Y_3)$;

2. As a function of γ it is a continuous curve in this Hilbert space and $\max\limits_{0\leq\gamma\leq 1} \|\delta(\gamma, Y)\|_{L^2} < \infty$.

Define the linearized map L_θ corresponding to θ as follows:

1. for $\gamma_{j+1} \leq \gamma \leq \gamma_j$, $j = 1, 2, \ldots$

$$L_\theta(\delta(\gamma, Y)) = \delta(\gamma(1+\theta), Y);$$

2. for $\dfrac{1}{1+\theta} \leq \gamma \leq 1$ the function $L_\theta(\delta(\gamma, Y))$ is given by the formula

$$L_\theta(\delta(\gamma, Y)) = \delta_{p_1}(1, Y, s)$$

where p_1 is found from the relation $\dfrac{p_1}{p} = \gamma$.

In other words at $\gamma = 1$ we use (32) to find the new $\delta^{(p+1)}(1, Y, s)$. After that we apply 1.

It is easy to see that there exist the limits $\lim\limits_{\substack{\theta\to 0\\ n\theta\to t}} L_\theta^n = A^t$ and the operators A^t constitute a semi-group. For $\gamma < 1$, $t > 0$ such that $\gamma e^t < 1$

$$A^t \delta(\gamma, Y) = \delta(\gamma e^t, Y)$$

Let $\mathcal{A}$ be the infinitesimal generator of the semi-group A^t. In §6 we study in more detail the spectrum and eigenfunctions of $\mathcal{A}$.

Lemma 5.1. *The eigenfunctions of the group A^t have the form*

$$\delta(\gamma, Y) = \gamma^\alpha \tilde{\Phi}_\alpha(Y)$$

where $\tilde{\Phi}_\alpha$ is a function with values in C^3 satisfying (32).

In more detail, if we take $\delta(\gamma, Y) = \gamma^\alpha \tilde{\Phi}_\alpha(Y)$ and substitute it into the rhs of (32) we get in the lhs $\delta^{(p+1)}(Y) = \tilde{\Phi}_\alpha(Y)$.

Proof. If $\delta(\gamma, Y)$ is an eigenfunction then from the formula for A^t

$$A^t\delta(\gamma, Y) = \delta(\gamma e^{-t}, Y) = e^{-\alpha t}\delta(\gamma, Y)$$

Let $\gamma \to 1$. Then

$$\delta(e^{-t}, Y) = e^{-\alpha t}\Phi(Y) = \gamma^\alpha \Phi(Y)$$

Lemma is proven.

The space Δ is spanned by the eigenfunctions of $\{A^t\}$ in the sense that for any $h \in \Delta$ we have the expansion

$$h(\gamma, Y) = \sum_{\alpha \in \mathsf{spec}\mathcal{A}} C^{(\alpha)}\gamma^\alpha \Phi_\alpha(Y)$$

The coefficients $C^{(\alpha)}$ are found with the help of the eigenfunctions of the conjugate system $\{(A^*)^t\}$. The form of the conjugate semi-group and its eigenfunctions can be investigated using the described above discrete approximation. We do not dwell more on this.

§6. The Spectrum of the Group of Linearized Maps

In this section we show that the solutions of (21) studied in §4 have $l^{(u)} = 4$ unstable eigenvalues and $l^{(n)} = 6$ neutral eigenvalues. Therefore in the renormalization group approach we consider 10– parameter families of initial conditions (see below).

As was already mentioned, in the limit $p \longrightarrow \infty$ the linearized maps generate a semi-group of operators acting in the space Δ of functions $f^{(j)}(\gamma, Y)$, $0 \leq \gamma \leq 1$, $Y \in \mathbb{R}^3$, $j = 1, 2$ which are continuous as functions of γ in the Hilbert space L^2. At $\gamma = 1$, the functions $f^{(j)}(\gamma, Y)$ satisfy the boundary condition which follows from (32):

$$\exp\left\{-\frac{\sigma^{(1)}}{2}\left(|Y_1|^2 + |Y_2|^2\right) - \frac{\sigma^{(2)}}{2}|Y_3|^2\right\} \cdot \frac{\sigma^{(1)}}{2\pi}\sqrt{\frac{\sigma^{(2)}}{2\pi}} \cdot f^{(j)}(1, Y)$$

$$= \int_0^1 d\gamma \int_{\mathbb{R}^3} \frac{\sigma^{(1)}}{2\pi\gamma} \cdot \sqrt{\frac{\sigma^{(2)}}{2\pi\gamma}} \cdot \frac{\sigma^{(1)}}{2\pi(1-\gamma)} \cdot \sqrt{\frac{\sigma^{(2)}}{2\pi(1-\gamma)}} \cdot$$

$$
\begin{aligned}
&\exp\left\{-\frac{\sigma^{(1)}(|Y_1-Y_1'|^2+|Y_2-Y_2'|^2)}{2\pi\gamma}-\frac{\sigma^{(2)}|Y_3-Y_3'|^2}{2\pi\gamma}-\frac{\sigma^{(1)}(|Y_1'|^2+|Y_2'|^2)}{2\pi(1-\gamma)}-\right.\\
&\left.-\frac{\sigma^{(2)}|Y_3'|^2}{2\pi(1-\gamma)}\right\}\left\{\left[-(1-\gamma)^{\frac{3}{2}}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}\;H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)+\frac{Y_2-Y_2'}{\sqrt{\gamma}}\,H_2\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right.\right.\\
&\left.+\gamma^{\frac{1}{2}}(1-\gamma)\left(\frac{Y_1'}{\sqrt{1-\gamma}}\,H_1\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)+\frac{Y_2'}{\sqrt{1-\gamma}}\,H_2\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)\right]f^{(j)}\left(1-\gamma,\,\frac{Y'}{\sqrt{1-\gamma}}\right)\\
&+\left[-\left(1-\gamma\right)^{\frac{3}{2}}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}\,f^{(1)}\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}}\right)+\frac{Y_2-Y_2'}{\sqrt{\gamma}}\,f^{(2)}\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right.\\
&\left.+\gamma^{\frac{1}{2}}\,(1-\gamma)\left(\frac{Y_1'}{\sqrt{1-\gamma}}\,f^{(1)}\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}}\right)+\frac{Y_2'}{\sqrt{1-\gamma}}\,f^{(2)}\left(\gamma,\frac{Y-Y'}{\sqrt{\gamma}}\right)\right)\right)\right]\\
&\left.H_j\left(\frac{Y'}{\sqrt{1-\gamma}}\right)\right\}\,d^3Y',\qquad\qquad j\;=\;1,2
\end{aligned}\tag{33}
$$

Denote by $\mathcal{R}_p$ the linear operator which transforms $\{\delta^{(p)}(\gamma,Y,s)\}$ into $\{\delta^{(p+1)}(\gamma,Y,s)\}$. Here s is a parameter which plays no role in this section. As it was explained in §5, for each t there exists the limit $\lim\limits_{p\longrightarrow\infty}\mathcal{R}_p^{tp}=A^t$ so that the operators A^t constitute a semi-group having an infinitesimal generator $\mathcal{A}=\lim\limits_{t\downarrow 0}\frac{A^t-I}{t}$. In our case $\mathcal{A}\delta(\gamma,Y,s)\;=\;\gamma\,\frac{\partial\delta(\gamma,Y,s)}{\partial\gamma}$, $0<\gamma<1$ and for $\gamma\,=\,1$ the function $\delta(1,Y,s)$ satisfies the boundary condition (33) in which $f(\gamma,Y)\,=\,\delta^{(p+1)}(1,Y,s)$.

If α is an eigenvalue of $\mathcal{A}$, then the corresponding eigenfunction has the form $\gamma^{\alpha}\Phi_{\alpha,\sigma^{(1)},\sigma^{(2)}}(Y)$ (see Lemma 5.1), where $\Phi_{\alpha,\sigma^{(1)},\sigma^{(2)}}(Y)$ satisfies the equation (33) with $f(\gamma,Y)=\gamma^{\alpha}\Phi_{\alpha,\sigma^{(1)},\sigma^{(2)}}(Y)$. If $\Re(\alpha)>0$ ($\Re(\alpha)=0$) then the corresponding eigenvalue is called unstable (neutral). All other eigenvalues are called stable. The subspaces generated by unstable, neutral, stable eigenvalues are denoted by $\Gamma^{(u)}$, $\Gamma^{(n)}$, $\Gamma^{(s)}$ respectively.

As before, for $\Phi^{(j)}_{\alpha,\sigma^{(1)},\sigma^{(2)}}(Y)$ the following scaling relation with respect to $\sigma^{(1)},\sigma^{(2)}$ is valid:

$$
\Phi^{(j)}_{\alpha,\sigma^{(1)},\sigma^{(2)}}(Y)\;\propto\;\Phi^{(j)}_{\alpha,1,1}\,(\sqrt{\sigma^{(1)}}Y_1,\sqrt{\sigma^{(1)}}Y_2,\sqrt{\sigma^{(2)}}Y_3)
$$

Therefore it is enough to consider the above equation (33) for $\sigma^{(1)}=\sigma^{(2)}=1$. We again use the expansion over *Hermite* polynomials:

$$\Phi^{(j)}_{\alpha,1,1}(Y) \;=\; \Phi^{(j)}_{\alpha}(Y) \;=\; \sum_{m_1,\, m_2,\, m_3} f^{(j)}_{\alpha}(m_1, m_2, m_3)\, He_{m_1}(Y_1)\, He_{m_2}(Y_2) He_{m_3}(Y_3)$$

Here j takes values $1, 2, 3$. Since in m_3 it is the usual convolution and H does not depend on Y_3, it is enough to look for solutions of (33) having the form $f_{m_1,m_2}\delta_{m_3}$. Put $\beta = \alpha + \frac{m_3}{2}$ and $f^{(j)}_{\beta}(m_1, m_2) = f^{(j)}_{\alpha}(m_1, m_2)\delta_{m_3}$. We come to the linear system of recurrent relations

$$\begin{aligned}
f^{(j)}_{\beta}(m_1, m_2) = \sum_{\substack{m_1' + m_1'' = m_1 \\ m_2' + m_2'' = m_2}} & J^{(1)}_{m', m''+2\beta} \left((B_1 h^{(1)})\, (m_1', m_2') \;+ (B_2 h^{(2)})\, (m_1', m_2') \right) f^{(j)}_{\beta}(m_1'', m_2'') \\
+ \quad & J^{(2)}_{m', m''+2\beta} \cdot h^{(1)}(m_1', m_2') \cdot (B_1 f^{(j)}_{\beta})(m_1'', m_2'') \\
+ \quad & J^{(2)}_{m', m''+2\beta} \cdot h^{(2)}(m_1', m_2') \cdot (B_2 f^{(j)}_{\beta})(m_1'', m_2'') \\
+ \quad & J^{(1)}_{m'+2\beta, m''} \cdot \Big((B_1 f^{(1)}_{\beta})(m_1', m_2') + (B_2 f^{(2)}_{\beta})(m_1', m_2') \Big) h^{(j)}(m_1'', m_2'') \qquad (34) \\
+ \quad & J^{(2)}_{m'+2\beta, m''} \cdot f^{(1)}_{\beta}(m_1', m_2') \cdot (B_1 h^{(j)})(m_1'', m_2'') \\
+ \quad & J^{(2)}_{m'+2\beta, m''} \cdot f^{(2)}_{\beta}(m_1', m_2') \cdot (B_2 h^{(j)})(m_1'', m_2'')
\end{aligned}$$

Introduce the vector

$$f^{(d)}_{\beta} \;=\; \left(f^{(1)}_{\beta}\,(0, d),\; f^{(1)}_{\beta}\,(1, d-1)\,, \ldots\,, f^{(1)}_{\beta}\,(d, 0)\; f^{(2)}_{\beta}\,(0, d),\; f^{(2)}_{\beta}\,(1, d-1)\,, \ldots\,, f^{(2)}_{\beta}\,(d, 0) \right)^T$$

The vector $f^{(d)}_{\beta}$ contains all terms of degree d. The recurrent relation (34) can be written as

$$C^{(d)}_{\beta}\, f^{(d)}_{\beta} \;=\; b^{(d)}_{\beta}$$

where the vector $b^{(d)}_{\beta}$ contains terms of degree $\leq d-1$. Also $C^{(d)}_{\beta} \;\in\; \mathbb{R}^{2(d+1)\times 2(d+1)}$ is a matrix. Let $C^{(d)}_{\beta}(k, \ell)$ be its (k, ℓ)-entry. Then

$$
C_\beta^{(d)}(k,\ell) = \begin{cases} 1 - \dfrac{16d+32\beta-16+32k}{(d+2\beta+1)(d+2\beta+3)(d+2\beta+5)}, & \text{if } 1 \le k = \ell \le d+1 \\[2ex] 1 - \dfrac{80d+160\beta+80-32k}{(d+2\beta+1)(d+2\beta+3)(d+2\beta+5)}, & \text{if } d+2 \le k = \ell \le 2d+2 \\[2ex] -\dfrac{32(d+2\beta-k+2)}{(d+2\beta+1)(d+2\beta+3)(d+2\beta+5)}, & \text{if } 2 \le k \le d+1, \ell = d+k \\[2ex] -\dfrac{32(k-d-2\beta-1)}{(d+2\beta+1)(d+2\beta+3)(d+2\beta+5)}, & \text{if } d+2 \le k \le 2d+1, \ell = k-d \\[2ex] 0, & \text{all other cases} \end{cases}
$$

Note that $d + 2\Re(\beta) > -1$.

Lemma 6.1. *Assume $\Re(\beta) \ge 0$. There exists an integer $d_* > 0$, independent of β, such that for all $d \ge d_*$, the matrix $C_\beta^{(d)}$ is invertible.*

Proof. As d tends to infinity, $C_\beta^{(d)}$ tends to the identity matrix if $\Re(\beta) \ge 0$. A simple estimate on the diagonal and off-diagonal entries shows that for all β such that $\Re(\beta) \ge 0$ and sufficiently large d, the matrix $C_\beta^{(d)}$ becomes diagonally dominant. This implies the existence of the needed d_* and its independence of β.

A similar statement holds if we assume that $\Re(\beta) \ge -A$ for any given $A \le 0$. We formulate it as the following lemma.

Lemma 6.1′. *For any $A \ge 0$, there exists an integer $d_*(A) > 0$ which depends only on A, such that for all $d \ge d_*(A)$ and all β with $\Re(\beta) \ge -A$, the matrix $C_\beta^{(d)}$ is invertible.*

By Lemma 6.1, to find all eigen-values of $\mathcal{A}$ it amounts to solve the equation $\det(C_\beta^{(d)}) = 0$. The eigenvalue α is then found from the relation $\beta = \alpha + \frac{m_3}{2}$. Let

$$
a_1 = \left(1 - \frac{16}{(d+2\beta+3)(d+2\beta+5)}\right) \Big/ \left(\frac{32}{(d+2\beta+1)(d+2\beta+3)(d+2\beta+5)}\right)
$$

Then a_1 is the eigen-value of the matrix $\tilde{C}^{(d)} \in \mathbb{R}^{2(d+1)\times 2(d+1)}$ given by:

$$\tilde{C}^{(d)}(k,\ell) \;=\; \begin{cases} k-1, & \text{if } 1 \le k \;=\; \ell \le d+1 \\ 2d+2-k, & \text{if } d+2 \le k \;=\; \ell \le 2d+2 \\ d+2-k, & \text{if } 2 \le k \le d+1, \ell = d+k \\ k-d-1 & \text{if } d+2 \le k \le 2d+1, \ell = k-d \\ 0, & \text{all other cases} \end{cases}$$

It is not difficult to find that the eigen-values of $\tilde{C}^{(d)}$ are 0 and $d+1$ with algebraic multiplicity $d+2$ and d respectively. Solve the equations $a_1 = 0$ or $a_1 = d+1$ and use the condition $d + 2\Re(\beta) > -1$. The possible values of β are then given by

$$\beta = \frac{3-d}{2} \quad \text{or} \quad \frac{\sqrt{17}-4-d}{2}, \quad d = 1,\, 2,\, 3,\, \cdots$$

This fact immediately gives the following lemma.

Lemma 6.2. *Let $(\tilde{C}_\beta^{(d)})^{-1}$ be the inverse matrix of $\tilde{C}_\beta^{(d)}$ for $d \ge d^*(\beta)$, where $d^*(\beta) = 3-2\beta$ or $\sqrt{17}-4-2\beta$ is an integer. Then there exists an absolute constant $C_2 > 0$ such that for all $d \ge d^*(\beta)$*

$$\| \, (\tilde{C}_\beta^{(d)})^{-1} \, \| \le \, C_2 \, .$$

We now state our theorem about the properties of the solutions to the recurrent relation (34).

Theorem 6.3. *The only possible values of β for which (34) have nonzero solutions $f_\beta^{(j)}(m_1,m_2)$ is given by:*

$$\beta = \frac{3-m}{2} \quad or \quad \frac{\sqrt{17}-4-m}{2}, \quad m = 1,\, 2,\, 3,\, \cdots$$

The corresponding solutions $f_\beta^{(j)}(m_1,m_2)$ have the following property:

a) $\beta = (\sqrt{17}-4-m)/2$. In this case $f_\beta^{(j)}(m_1,m_2) = 0$ for any $0 \le m_1+m_2 < m$. For $d = m$, we have

$$f_\beta^{(1)}(r,d-r) = -(d-r+1) f_\beta^{(2)}(r-1,d-r+1), \quad r = 1,2,\cdots,d$$

$f_\beta^{(1)}(0,d)$, $f_\beta^{(2)}(d,0)$ are free parameters. $f_\beta^{(j)}(m_1,m_2)$ for $m_1+m_2 \geq m+1$ are uniquely determined if the values of the $m+2$ free parameters $f_\beta^{(1)}(r,m-r), r=0,1,\cdots,m$, and $f_\beta^{(2)}(m,0)$ are specified.

b) $\beta=(3-m)/2$. In this case $f_\beta^{(j)}(m_1,m_2)=0$ for any $0 \leq m_1+m_2 < m$. For $d=m$, we have $f_\beta^{(1)}(0,d)=f_\beta^{(2)}(d,0)=0$, and

$$f_\beta^{(1)}(r,d-r)=f_\beta^{(2)}(r-1,d-r+1), \quad r=1,\cdots,d$$

are free parameters. $f_\beta^{(j)}(m_1,m_2)$ for $m_1+m_2 \geq m+1$ are uniquely determined if the values of the m free parameters $f_\beta^{(1)}(r,m-r), r=1,\cdots,m$ are specified.

In both case a) and b), the solutions $f_\beta^{(j)}(m_1,m_2)$ is zero for $m_1+m_2=m+1,m+3,\ldots$. Since $f_\beta^{(j)}$ depends linearly on the free parameters, we have for some $C_3>0$, $0<\rho<\frac{1}{4000}$

$$\left|f_\beta^{(j)}(m_1,m_2)\right| \leq C_3\,\frac{\rho^{m_1+m_2+2\beta}}{\Gamma\left(\frac{m_1+m_2+2\beta+3}{2}\right)}, \quad \forall\, m_1 \geq 0,\; m_2 \;\geq\; 0,\; j=1,2.$$

Proof. Property a) and b) are straightforward computations. From recurrent relation (34), by parity it is obvious that $f_\beta^{(j)}(m_1,m_2)=0$ for $m_1+m_2=m+1$. An easy induction shows that $f_\beta^{(j)}(m_1,m_2)=0$ for $m_1+m_2=m+3,m+5,\ldots$ We now prove the decay estimate. The strategy of the proof is the same as in theorem 4.3. From the proof of theorem 4.3, it is clear that by choosing the parameters $(x^{(1)},x^{(2)},x^{(3)})$ sufficiently small, we have

$$\left|h_{m_1,m_2}^{(j)}\right| \leq \frac{\rho^{m_1+m_2+2}}{\Gamma\left(\frac{m_1+m_2+7}{2}\right)} \quad \forall\, m_1 \geq 0,\; m_2 \;\geq\; 0, m_1+m_2 \geq 3,\; j=1,2.$$

Our induction hypothesis for $f_\beta^{(j)}(m_1,m_2)$ is

$$\left|f_\beta^{(j)}(m_1,m_2)\right| \leq \frac{\rho^{m_1+m_2+2\beta}}{\Gamma\left(\frac{m_1+m_2+2\beta+3}{2}\right)} \quad \forall\, m \leq m_1+m_2 < d,\; j=1,2.$$

where $d \geq L$ and $d-m$ is an even number (note that $f_\beta^{(j)}(m_1,m_2)=0$ for $m_1+m_2=m+1,m+3,\ldots$). We assume that L is a sufficiently large number and will verify the

induction assumption for $m \le d \le L$ later. Now for $m_1 + m_2 = d$, by lemma 6.2, we have

$$\begin{aligned}
\left|f_\beta^{(j)}(m_1, m_2)\right| \le & C_2 \cdot \sum_{m'=2}^{d-m} (m'+1) \cdot \left|J^{(1)}_{m',m''+2\beta}\right| \cdot 2(m'+1) \cdot \frac{\rho^{m'+3}}{\Gamma\left(\frac{m'+8}{2}\right)} \cdot \frac{\rho^{m''+2\beta}}{\Gamma\left(\frac{m''+2\beta+3}{2}\right)} \\
& + C_2 \cdot \sum_{m'=4}^{d-m} (m'+1) \cdot \left|J^{(1)}_{m',m''+2\beta}\right| \cdot 2 \cdot \frac{\rho^{m'+1}}{\Gamma\left(\frac{m'+6}{2}\right)} \cdot \frac{\rho^{m''+2\beta}}{\Gamma\left(\frac{m''+2\beta+3}{2}\right)} \\
& + C_2 \cdot \left|J^{(1)}_{2,d-2+2\beta}\right| \cdot 4 \cdot \frac{\rho^{d-2+2\beta}}{\Gamma\left(\frac{d+2\beta+1}{2}\right)} \\
& + C_2 \cdot \sum_{m'=3}^{d-m+1} (m'+1) \cdot \left|J^{(2)}_{m',m''+2\beta}\right| \cdot 2 \cdot \frac{\rho^{m'+2}}{\Gamma\left(\frac{m'+7}{2}\right)} \cdot (m''+1) \cdot \frac{\rho^{m''+2\beta+1}}{\Gamma\left(\frac{m''+2\beta+4}{2}\right)} \\
& + C_2 \cdot \sum_{m'=3}^{d-m-1} (m'+1) \cdot \left|J^{(2)}_{m',m''+2\beta}\right| \cdot 2 \cdot \frac{\rho^{m'+2}}{\Gamma\left(\frac{m'+7}{2}\right)} \cdot \frac{\rho^{m''+2\beta-1}}{\Gamma\left(\frac{m''+2\beta+2}{2}\right)} \\
& + C_2 \cdot \left|J^{(2)}_{1,d-1+2\beta}\right| \cdot 4 \cdot \frac{\rho^{d-1+2\beta}}{\Gamma\left(\frac{d+2\beta+2}{2}\right)} \\
& + C_2 \cdot \sum_{m'=m-1}^{d-3} (m''+1) \cdot \left|J^{(1)}_{m'+2\beta,m''}\right| \cdot 2 \cdot (m'+1) \cdot \frac{\rho^{m'+2\beta+1}}{\Gamma\left(\frac{m'+2\beta+4}{2}\right)} \cdot \frac{\rho^{m''+2}}{\Gamma\left(\frac{m''+7}{2}\right)} \\
& + C_2 \cdot \sum_{m'=m+1}^{d-3} (m''+1) \cdot \left|J^{(1)}_{m'+2\beta,m''}\right| \cdot 2 \cdot \frac{\rho^{m'+2\beta-1}}{\Gamma\left(\frac{m'+2\beta+3}{2}\right)} \cdot \frac{\rho^{m''+2}}{\Gamma\left(\frac{m''+7}{2}\right)} \\
& + C_2 \cdot \left|J^{(1)}_{d-1+2\beta,1}\right| \cdot 4 \cdot \frac{\rho^{d-2+2\beta}}{\Gamma\left(\frac{d+2\beta+1}{2}\right)} \\
& + C_2 \cdot \sum_{m'=m}^{d-2} (m''+1) \cdot \left|J^{(2)}_{m'+2\beta,m''}\right| \cdot 2 \cdot \frac{\rho^{m'+2\beta}}{\Gamma\left(\frac{m'+2\beta+3}{2}\right)} \cdot \frac{\rho^{m''+3}}{\Gamma\left(\frac{m''+8}{2}\right)} \cdot (m''+1) \\
& + C_2 \cdot \sum_{m'=m}^{d-4} (m''+1) \cdot \left|J^{(2)}_{m'+2\beta,m''}\right| \cdot 2 \cdot \frac{\rho^{m'+2\beta}}{\Gamma\left(\frac{m'+2\beta+3}{2}\right)} \cdot \frac{\rho^{m''+1}}{\Gamma\left(\frac{m''+6}{2}\right)} \\
& + C_2 \cdot \left|J^{(2)}_{d-2+2\beta,2}\right| \cdot 4 \cdot \frac{\rho^{d-2+2\beta}}{\Gamma\left(\frac{d+2\beta+1}{2}\right)}
\end{aligned}$$

$$
\begin{aligned}
\le & C_2 \cdot \frac{\rho^{d+2\beta}}{\Gamma\left(\frac{d+2\beta+3}{2}\right)} \cdot \left(\sum_{m'=2}^{d-m} \frac{8\rho^3}{d+2\beta+5} + \sum_{m'=4}^{d-m} \frac{8\rho}{d+2\beta+5} + \frac{8}{\rho^2 \cdot (d+2\beta+5)} \right. \\
& + \sum_{m'=3}^{d-m+1} \frac{8\rho^3}{d+2\beta+5} \cdot \frac{2(m''+1)}{d+2\beta+3} + \sum_{m'=3}^{d-m-1} \frac{8\rho}{d+2\beta+5} + \frac{8}{\rho(d+2\beta+5)} \\
& + \sum_{m'=m-1}^{d-3} \frac{8\rho^3}{d+2\beta+5} \cdot \frac{2(m'+1)}{d+2\beta+3} + + \sum_{m'=m+1}^{d-3} \frac{8\rho}{d+2\beta+5} + \frac{16}{\rho^2(d+2\beta+5)} \\
& \left. + \sum_{m'=m}^{d-2} \frac{16\rho^3}{d+2\beta+5} + \sum_{m'=m}^{d-4} \frac{8\rho}{d+2\beta+5} + \frac{16}{\rho^2(d+2\beta+5)} \right) \\
& \le C_2 \cdot \frac{\rho^{d+2\beta}}{\Gamma\left(\frac{d+2\beta+3}{2}\right)} \cdot 2000\rho \le \frac{\rho^{d+2\beta}}{\Gamma\left(\frac{d+2\beta+3}{2}\right)}
\end{aligned}
$$

where in the second last inequality above we have used the fact that $d \ge L$ and L is sufficiently large such that $d/(d+2\beta+3) \le 2$. It is clear that it suffices for us to take $L = 2m$. To check the inductive assumption for $m \le d \le 2m$, we recall that $f_\beta^{(j)}(m_1, m_2)$ depends linearly on several free parameters. If we let them be sufficiently small, then it is clear that the inductive assumption is satisfied for $m \le d \le 2m$. Our theorem is proved.

We now formulate our main theorem about the spectrum of the linearized operator.

Theorem 6.4. *The spectrum of the operator $\mathcal{A}$ consists of the following eigen-values*

$$
\mathsf{spec}\,(\mathcal{A}) = \left\{ 1, \frac{1}{2}, 0, \lambda_m^{(1)}, \lambda_m^{(2)}, m \ge 1 \right\}.
$$

where $\lambda_m^{(1)} = -\frac{m}{2}$, $\lambda_m^{(2)} = \frac{\sqrt{17}-4-m}{2}$, $m \ge 1$.

The first eigen-values have multiplicities $\nu_1 = 1$, $\nu_{\frac{1}{2}} = 3$, $\nu_0 = 6$. The eigen-values $\lambda_m^{(1)}$, $\lambda_m^{(2)}$ correspond to the stable part of the spectrum and also have finite multiplicities given by: $\nu_{\lambda_m^{(1)}} = \frac{(m+3)(m+4)}{2}$, $\nu_{\lambda_m^{(2)}} = \frac{m(m+5)}{2}$.

For each $\alpha \in \mathsf{spec}\,(\mathcal{A})$, the eigenfunctions $f_\alpha^{(j)}(m_1, m_2, m_3)$ have the following property:

a) $f_\alpha^{(j)}(m_1, m_2, m_3)$ is compactly supported in the m_3 variable, i.e., there exists an integer $m_3^* = m_3^*(\alpha)$ such that

$$
f_\alpha^{(j)}(m_1, m_2, m_3) = 0 \quad \textit{if } m_3 > m_3^*
$$

b) $f_\alpha^{(j)}(m_1, m_2, m_3)$ *decays faster than exponentially, more precisely, there exist constants* $C_3 = C_3(\alpha) > 0$ *and* $0 < \rho < \frac{1}{4000}$*, such that*

$$\left|f_\alpha^{(j)}(m_1, m_2, m_3)\right| \le C_3 \frac{\rho^{m_1+m_2+m_3+2\alpha}}{\Gamma\left(\frac{m_1+m_2+m_3+2\alpha+3}{2}\right)}, \quad \forall\ m_1, m_2, m_3 \ge 0$$

The system of eigenfunctions is complete in the following sense. Let $\Gamma^{(s)}$ *be the stable linear subspace of* Δ *generated by all eigenfunctions with* $\Re(\lambda) < 0$*,* $\Gamma^{(u)}$ *be the unstable subspace generated by all eigenfunctions with eigenvalues* $\lambda > 0$*, and* $\Gamma^{(n)}$ *be the neutral subspace generated by all eigenfunctions with eigenvalue* $\lambda = 0$*. Then* $\dim \Gamma^{(u)} = 4$*,* $\dim \Gamma^{(n)} = 6$ *and*

$$\Delta \ = \ \Gamma^{(u)} \ + \ \Gamma^{(n)} \ + \ \Gamma^{(s)} \, .$$

Proof. By Lemma 6.1, we only need to examine β for which $\det(C_\beta^{(d)}) = 0$. From previous arguments, we have that for $d \ge 1$, $\beta = -\frac{d-3}{2}$ or $\frac{\sqrt{17}-4-d}{2}$. We discuss the spectrum separately in the following three cases.

<u>1° unstable spectrum:</u> $\quad \alpha \ = \ 1, 1/2$.

a) $\alpha \ = 1$. Since $\beta = \alpha + \frac{m_3}{2}$, the only possibility is that $\beta = 1$, $d = 1$ and $m_3 = 0$. The eigenspace is one-dimensional with $f_{000}^{(1)} = f_{000}^{(2)} = f_{010}^{(1)} = f_{100}^{(2)} = 0$, $f_{100}^{(1)} = f_{010}^{(2)}$ is a free parameter and the remaining part of all higher degree terms ($f_{m_1,m_2,0}^{(j)}$ with $m_1 + m_2 \ge 2$) is uniquely determined once we specify $f_{100}^{(1)}$.

b) $\alpha = 1/2$. Possible cases are $m_3 = 0$, $\beta = 1/2$, $d = 0$, 2 or $m_3 = 1$, $\beta = 1$, $d = 1$. In the first case we have $f_{m_1,m_2,0}^{(j)} = 0$ for $m_1 + m_2 \le 1$, $f_{110}^{(1)} = f_{020}^{(2)}$, $f_{200}^{(1)} = f_{110}^{(2)}$ are two free parameters, all other terms of higher degee ($f_{m_1,m_2,0}^{(j)}$ with $m_1 + m_2 \ge 3$) are uniquely determined once we specify the above four parameters. In the second case we have $f_{001}^{(1)} = f_{001}^{(2)} = f_{011}^{(1)} = f_{101}^{(2)} = 0$, $f_{101}^{(1)} = f_{011}^{(2)}$ is a free parameter and the remaining part of all higher degree terms ($f_{m_1,m_2,1}^{(j)}$ with $m_1 + m_2 \ge 2$) is uniquely determined once we specify $f_{101}^{(1)}$. Putting two cases together, we see that the dimension of the eigenspace is 3.

This gives $\dim \Gamma^{(u)} = 4$.

<u>2° neutral spectrum:</u> Here we have $\alpha = 0$, and three cases.

a) $m_3 = 2$. Then $\beta = 1$. The eigenspace is one-dimensional with $f^{(1)}_{002} = f^{(2)}_{002} = f^{(1)}_{012} = f^{(2)}_{102} = 0$, $f^{(1)}_{102} = f^{(2)}_{012}$ is a free parameter and the remaining part of all higher degree terms ($f^{(j)}_{m_1,m_2,2}$ with $m_1 + m_2 \geq 2$) is uniquely determined once we specify $f^{(1)}_{102}$. This eigenvector is connected with $\frac{\partial}{\partial \sigma^{(2)}}$ which corresponds to the variation of the parameter $\sigma^{(2)}$ of the fixed point.

b) $m_3 = 1$. Then $\beta = 1/2$. We have $f^{(j)}_{m_1,m_2,1} = 0$ for $m_1 + m_2 \leq 1$, $f^{(1)}_{111} = f^{(2)}_{021}$, $f^{(1)}_{201} = f^{(2)}_{111}$ are two free parameters, all other terms of higher degee ($f^{(j)}_{m_1,m_2,1}$ with $m_1 + m_2 \geq 3$) are uniquely determined once we specify the above two parameters. Clearly the eigenspace is two-dimensional. This eigenspace does not correspond to any change of parameters of the fixed point.

c) $m_3 = 0$. Then $\beta = 0$. We have $f^{(j)}_{m_1,m_2,0} = 0$ for $m_1 + m_2 \leq 2$, $f^{(1)}_{030} = f^{(2)}_{300} = 0$, $f^{(1)}_{120} = f^{(2)}_{030}$, $f^{(1)}_{210} = f^{(2)}_{120}$, $f^{(1)}_{300} = f^{(2)}_{210}$ are three free parameters. All other terms of higher degee ($f^{(j)}_{m_1,m_2,0}$ with $m_1 + m_2 \geq 4$) are uniquely determined once we specify the above three parameters. This eigenspace corresponds to $(\frac{\partial}{\partial x^{(1)}}, \frac{\partial}{\partial x^{(2)}}, \frac{\partial}{\partial x^{(3)}})$.

Putting all three cases together, we see that $\dim \Gamma^{(n)} = 6$.

<u>3° stable spectrum:</u> $\Re(\alpha) < 0$.

There are two cases.

<u>Case 1</u>: $\alpha = -\frac{m}{2}$, $m \geq 1$. Recall that $\beta = \alpha + \frac{m_3}{2}$, and m_3 satisfies $0 \leq m_3 \leq m + 2$. By theorem 6.3, for each such β, the number of free parameters is $3 - 2\beta$. Then the total multiplicity ν_α is given by

$$\nu_\alpha = \sum_{m_3=0}^{m+2} 3 - (-m + m_3) = \frac{(m+3)(m+4)}{2}$$

Case 2: $\alpha = \frac{\sqrt{17}-4-m}{2}$, $m \geq 1$. $\beta = \alpha + \frac{m_3}{2}$, and m_3 satisfies $0 \leq m_3 \leq m-1$. By theorem 6.3, we have

$$\nu_\alpha = \sum_{m_3=0}^{m-1} (m - m_3 + 2) = \frac{m(m+5)}{2}$$

It follows easily that the eigenfunctions $f_\alpha^{(j)}(m_1, m_2, m_3)$ is compactly supported in the m_3 variable. By theorem 6.3, the decay estimate on $f_\alpha^{(j)}(m_1, m_2, m_3)$ is obvious.

It turns out that the eigenvector corresponding to $\frac{\partial}{\partial \sigma^{(1)}}$ is in the eigenspace spanned by the eigenvectors $(\frac{\partial}{\partial x^{(1)}}, \frac{\partial}{\partial x^{(2)}}, \frac{\partial}{\partial x^{(3)}})$. More precisely we have the following:

Lemma 6.3. *Let $t_1 = x^{(1)} - 1$, $t_2 = x^{(2)}$, $t_3 = x^{(3)} - 1$. Then*

$$\tilde{G}^{(\sigma^{(1)}, t_1, t_2, t_3, \sigma^{(2)})}(Y) = G^{(\sigma^{(1)}, x^{(1)}, x^{(2)}, x^{(3)}, \sigma^{(2)})}(Y) \tag{34}$$

where $G^{(\sigma^{(1)}, x^{(1)}, x^{(2)}, x^{(3)}, \sigma^{(2)})}$ is defined in (30). The function $\tilde{G}$ satisfies the following scaling relation:

$$\tilde{G}^{(\sigma^{(1)}, t_1, t_2, t_3, \sigma^{(2)})}(Y) = \tilde{G}^{(1, \sigma^{(1)} t_1, \sigma^{(1)} t_2, \sigma^{(1)} t_3, \sigma^{(2)})}(Y) \tag{35}$$

Proof. Let $f_{m_1, m_2, 0}^{(j),0}$ correspond to the eigenvector $\frac{\partial}{\partial \sigma^{(1)}}$, then a simple calculation shows that

$$f_{m_1,m_2,0}^{(j),0} = (m_1 + m_2 - 1)\, h_{m_1 m_2}^{(j)} + h_{m_1-2,m_2}^{(j)} + h_{m_1,m_2-2}^{(j)}.$$

If $f_{m_1,m_2,0}^{(j),1}$, $f_{m_1,m_2,0}^{(j),2}$ and $f_{m_1,m_2,0}^{(j),3}$ correspond to the eigenvectors $\frac{\partial}{\partial x^{(1)}}$, $\frac{\partial}{\partial x^{(1)}}$, and $\frac{\partial}{\partial x^{(3)}}$ respectively, then clearly we have

$$f_{m_1,m_2,0}^{(j),0} = \left(x^{(1)} - 1\right) f_{m_1,m_2,0}^{(j),1} + x^{(2)} f_{m_1,m_2,0}^{(j),2} + \left(x^{(3)} - 1\right) f_{m_1,m_2,0}^{(j),3}$$

This immediately gives

$$\left[\sigma^{(1)} \frac{\partial}{\partial \sigma^{(1)}} - \left(x^{(1)} - 1\right) \frac{\partial}{\partial x^{(1)}} - x^{(2)} \frac{\partial}{\partial x^{(2)}} - \left(x^{(3)} - 1\right) \frac{\partial}{\partial x^{(3)}}\right] G^{(\sigma^{(1)}, x^{(1)}, x^{(2)}, x^{(3)}, \sigma^{(2)})}(Y) = 0\,.$$

Regarding this as a transport equation in the variables $(\sigma^{(1)}, t_1, t_2, t_3)$, we can easily find that $\tilde{G}$ satisfies the scaling (35). Lemma is proved.

This lemma actually shows in what sense the parameters $\sigma^{(1)}, x^{(1)}, x^{(2)}, x^{(3)}$ are dependent.

As was shown in §4, we have the five-parameter family of fixed points $G^{(\sigma^{(1)}, x^{(1)}, x^{(2)}, x^{(3)}, \sigma^{(2)})}$. We use the notation $\pi = (\sigma^{(1)}, x^{(1)}, x^{(2)}, x^{(3)}, \sigma^{(2)})$ and write $G^{(\pi)}$ instead of $G^{(\sigma^{(1)}, x^{(1)}, x^{(2)}, x^{(3)}, \sigma^{(2)})}$. The spectrum of the linearization of the equation for the fixed point does not depend on π (see §5) and has $\ell^{(u)} = 4$ unstable eigenvectors $\Phi_j^{(u)}(Y_1, Y_2, Y_3)$, $1 \le j \le \ell^{(u)} = 4$ and $\ell^{(n)} = 6$ neutral eigenvectors $\Phi_{j'}^{(n)}(Y_1, Y_2, Y_3)$, $1 \le j' \le \ell^{(n)} = 6$.

§7. The Choice of Initial Conditions and the Initial Part of the Inductive Procedure

The equation (21) for the fixed point which was derived in §3 is non-typical from the point of view of the renormalization group theory because it contains the integration over γ, $0 \le \gamma \le 1$. On the other hand, since we consider the Cauchy problem for (1) we are given only the initial condition $v(k,0)$ which produces through the recurrent relations (4), (5), (6) or (4′), (5′), (6′) the whole set of functions $h_r(k,s)$ or $g_r(\tilde{k}, s)$. For large p and $r \le p$ they can be considered as depending on $\gamma = \frac{r}{p}$ and our procedure is organized in such a way that for γ which are away from zero $\tilde{g}_r$ are close to their limits. Therefore the initial part of our process should be discussed in more detail. This is done in this section.

We take $k^{(0)}$ which will be assumed to be sufficiently large, introduce the neighborhood

$$A_1 \, = \, \{k : \, \left|k - \kappa^{(0)}\right| \le D_1 \sqrt{k^{(0)} ln k^{(0)}}\}$$

where $\kappa^{(0)} = (0, 0, k^{(0)})$ and D_1 is also sufficiently large. Our initial conditions will be zero outside A_1. Inside A_1 they have the form

$$v(k,0) = \frac{1}{2\pi} \exp\left\{-\frac{Y_1^2 + Y_2^2}{2}\right\} \Bigg(H^{(0)}(Y_1, Y_2) + \sum_{j=1}^{4} b_j^{(u)} \Phi_j^{(u)}(Y_1, Y_2, Y_3) + \\ \sum_{j'=1}^{6} b_{j'}^{(n)} \Phi_{j'}^{(n)}(Y_1, Y_2, Y_3) + \Phi(Y_1, Y_2, Y_3; b^{(u)}, b^{(n)}) \Bigg) \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{Y_3^2}{2}\right\}$$

In this expression $k = k^{(0)} + \sqrt{k^{(0)}}Y$, $H^{(0)}(Y_1, Y_2) = (H_1^{(0)}(Y_1, Y_2), H_2^{(0)}(Y_1, Y_2), 0)$ is the fixed point of our renormalization group (see §4) corresponding to the parameters $\sigma_1^{(1)} = \sigma_1^{(2)} = 1$, $x_1 = x_2 = x_3 = 0$. Also $\Phi_j^{(u)}$, $\Phi_{j'}^{(n)}$ are unstable and neutral eigen-functions for $H^{(0)}$ described in §6, $b_j^{(u)}$ and $b_{j'}^{(n)}$ are our main parameters, $-\rho_1 \leq b_j^{(u)}, b_{j'}^{(n)} \leq \rho_1$ where ρ_1 is another constant which depends on $k^{(0)}$. Its value will also be specified later. Each function $\Phi(Y_1, Y_2, Y_3; b^{(u)}, b^{(n)})$, $b^{(u)} = \{b_j^{(u)}\}$, $b^{(n)} = \{b_{j'}^{(n)}\}$ is small in the sense that they satisfy

$$\sup_{Y,b} \left|\Phi(Y_1, Y_2, Y_3; b^{(u)}, b^{(n)})\right| \leq D_2,$$

$$\sup \left\|\Phi(Y_1, Y_2, Y_3; \bar{b}^{(u)}, \bar{b}^{(n)}) - \Phi(Y_1, Y_2, Y_3; b^{(u)}, b^{(n)})\right\| \leq D_2(\left|\bar{b}^{(u)} - b^{(u)}\right| + \left|\bar{b}^{(n)} - b^{(n)}\right|).$$

Due to the presence of $b^{(u)}$, $b^{(n)}$, we have $l = l^{(u)} + l^{(n)} = 10$-parameter families of initial conditions, due to the presence of Φ we have an open set in the space of such families.

Let

$$A_r = \{k : \ |k - r\kappa^{(0)}| \leq D_1\sqrt{rk^{(0)} \ln rk^{(0)}}\}$$

and the variable Y be such that $k = r\kappa^{(0)} + \sqrt{rk^{(0)}}Y$. Assume that for $r < p$, $|Y| \leq D_1\sqrt{\ln rk^{(0)}}$

$$h_r(r\kappa^{(0)} + \sqrt{rk^{(0)}}Y, s) = Z_p(s)\Lambda_p^{r-1}(s)r\tilde{g}_r(Y, s)$$

and

$$\begin{aligned}
\tilde{g}_r(Y, s) =& \frac{1}{2\pi} \exp\left\{-\frac{Y_1^2 + Y_2^2}{2}\right\} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{Y_3^2}{2}\right\} \cdot r \\
&\cdot \Bigg(H_1^{(0)}(Y_1, Y_2) + \delta_1^{(r)}(Y_1, Y_2, Y_3), H_2^{(0)}(Y_1, Y_2) + \delta_2^{(r)}(Y_1, Y_2, Y_3), \\
&\qquad \frac{1}{\sqrt{rk^{(0)}}}(F^{(r)}(Y_1, Y_2) + \delta_3^{(r)}(Y_1, Y_2, Y_3))\Bigg)
\end{aligned}$$

where in view of incompressibility

$$H_1^{(0)}Y_1 + H_2^{(0)}Y_2 + F^{(r)} = 0 \tag{36}$$

We shall derive a system of recurrent relations for $Z_p(s)$ and $\Lambda_p(s)$ for $p < p_0$. All $\delta_j^{(r)}$ will be considered as remainder terms.

Outside A_r we assume that

$$|h_r(r\kappa^{(0)} + \sqrt{rk^{(0)}}Y, s)| \le \frac{1}{(rk^{(0)})^{\lambda_1}}$$

where λ_1 is another constant which depends on C_1.

Returning back to (6) take the term with some p_1, p_2, $p_1 + p_2 = p$ and introduce the new variable of integration Y' where $k' = p_2\kappa^{(0)} + \sqrt{pk^{(0)}}Y'$. Introduce also the variables θ_1, θ_2, $0 \le \theta_1 \le (p_1k^{(0)})^2$, $0 \le \theta_2 \le (p_2k^{(0)})^2$ where $s_1 = s\left(1 - \frac{\theta_1}{(p_1k^{(0)})^2}\right)$, $s_2 = s\left(1 - \frac{\theta_2}{(p_2k^{(0)})^2}\right)$.

Then from (6)

$$
\begin{aligned}
&h_p(p\kappa^{(0)}+\sqrt{pk^{(0)}}Y,s)=Z_{p+1}(s)\Lambda_{p+1}^{p}(s)p\tilde{g}_p(Y,s)\\
&=(pk^{(0)})^{\frac{5}{2}}i\int\limits_0^{((p-1)k^{(0)})^2}d\theta_2\int\limits_{\mathbb{R}^3}\exp\left\{-\theta_2|\kappa^{(0,0)}+\frac{Y'}{\sqrt{(p-1)k^{(0)}}}|^2\right\}\cdot\\
&\qquad Z_p(s(1-\frac{\theta_2}{((p-1)k^{(0)})^2})\cdot\Lambda_p^{p-1}(s(1-\frac{\theta_2}{((p-1)k^{(0)})^2}))\cdot(p-1)\cdot Z_p(s)\Lambda_p(s)\\
&\left\langle\tilde{g}_1((Y-Y')\sqrt{pk^{(0)}},0),\kappa^{(0,0)}+\frac{Y}{\sqrt{pk^{(0)}}}\right\rangle P_{\kappa^{(0,0)}+\frac{Y}{\sqrt{pk^{(0)}}}}\tilde{g}_{p-1}\left(Y'\sqrt{\frac{p}{p-1}},s(1-\frac{\theta_2}{((p-1)k^{(0)})^2})\right)d^3Y'+\\
&+ip\sum_{\substack{p_1+p_2=p\\ p_1,p_2>1}}\frac{1}{p}\frac{(pk^{(0)})^{\frac{5}{2}}p_1p_2}{(p_1k^{(0)})^2(p_2k^{(0)})^2}\int\limits_0^{(p_1k^{(0)})^2}d\theta_1\int\limits_0^{(p_2k^{(0)})^2}d\theta_2\int\limits_{\mathbb{R}^3}\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\exp\left\{-\frac{(Y_1-Y_1')^2+(Y_2-Y_2')^2+(Y_3-Y_3')^2}{2\gamma}\right\}\\
&\left\langle\tilde{g}_{p_1}\left(\frac{Y-Y'}{\sqrt{\gamma}},s\left(1-\frac{\theta_1}{(p_1k^{(0)})^2}\right)\right),\kappa^{(0,0)}+\frac{Y}{\sqrt{pk^{(0)}}}\right\rangle P_{\kappa^{(0,0)}+\frac{Y}{\sqrt{pk^{(0)}}}}\tilde{g}_{p_2}\left(\frac{Y'}{\sqrt{1-\gamma}},s\left(1-\frac{\theta_2}{(p_2k^{(0)})^2}\right)\right)\\
&\qquad Z_p(s(1-\frac{\theta_1}{(p_1k^{(0)})^2}))\cdot\Lambda_p^{p_1-1}(s(1-\frac{\theta_1}{(p_1k^{(0)})^2}))\cdot Z_p(s(1-\frac{\theta_2}{(p_2k^{(0)})^2}))\cdot\Lambda_p^{p_2-1}(s(1-\frac{\theta_2}{(p_2k^{(0)})^2}))\cdot\\
&\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\exp\left\{-\frac{(Y_1')^2+(Y_2')^2+(Y_3')^2}{2(1-\gamma)}\right\}\exp\left\{-\theta_1|\kappa^{(0,0)}+\frac{Y-Y'}{\gamma\sqrt{pk^{(0)}}}|^2\right\}\exp\left\{-\theta_2|\kappa^{(0,0)}+\frac{Y'}{(1-\gamma)\sqrt{pk^{(0)}}}|^2\right\}+\\
&+\frac{i(pk^{(0)})^{\frac{5}{2}}(p-1)}{((p-1)k^{(0)})^2}\int\limits_0^{((p-1)k^{(0)})^2}d\theta_1\int\limits_{\mathbb{R}^3}\exp\left\{-\theta_1|\kappa^{(0,0)}+\frac{Y-Y'}{\sqrt{(p-1)k^{(0)}}}|^2\right\}\\
&\qquad Z_p(s(1-\frac{\theta_1}{((p-1)k^{(0)})^2})\cdot\Lambda_p^{p-1}(s(1-\frac{\theta_1}{((p-1)k^{(0)})^2}))\cdot Z_p(s)\Lambda_p(s)\\
&\left\langle\tilde{g}_{p-1}((Y-Y')\sqrt{\frac{p}{p-1}},s(1-\frac{\theta_1}{((p-1)k^{(0)})^2}),\kappa^{(0,0)}+\frac{Y-Y'}{\sqrt{pk^{(0)}}}\right\rangle P_{\kappa^{(0,0)}+\frac{Y}{\sqrt{pk^{(0)}}}}\tilde{g}_1\left(Y'\sqrt{p},s)\right)d^3Y'
\end{aligned}
\tag{37}
$$

Here $\gamma=\dfrac{p_1}{p}$ and $\kappa^{(0,0)}=(0,0,1)$. Now we shall modify (37) for $p_1>1$, $p_2>1$ similar to what we did in §3. Later we discuss the terms with $p_1=1$ and $p_2=1$. The modification consists of four steps.

Step 1. All terms $s\left(1-\frac{\theta_1}{(p_1k^{(0)})^2}\right)$,$s\left(1-\frac{\theta_2}{(p_2k^{(0)})^2}\right)$ are replaced by s.

Step 2. Write

$$\frac{(pk^{(0)})^{\frac{5}{2}}p_1p_2}{(p_1k^{(0)})^2(p_2k^{(0)})^2} = \frac{(pk^{(0)})^{\frac{1}{2}}}{(k^{(0)})^2\gamma(1-\gamma)}$$

Step 3. Consider the inner product

$$(pk^{(0)})^{\frac{1}{2}} \left\langle \tilde{g}_{p_1}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right), \kappa^{(0,0)} + \frac{Y}{\sqrt{pk^{(0)}}} \right\rangle$$

Up to remainders and from (36) it equals to

$$\left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \exp\left\{-\frac{(Y_1-Y_1')^2+(Y_2-Y_2')^2+(Y_3-Y_3')^2}{2\gamma}\right\}$$

$$\left[H_1^{(0)}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}, \frac{Y_2-Y_2'}{\sqrt{\gamma}}\right)Y_1 + H_2^{(0)}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}, \frac{Y_2-Y_2'}{\sqrt{\gamma}}\right)Y_2 + \frac{1}{\sqrt{\gamma}}F^{(p_1)}\left(\frac{Y-Y'}{\sqrt{\gamma}}, s\right)\right] =$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \exp\left\{-\frac{(Y_1-Y_1')^2+(Y_2-Y_2')^2+(Y_3-Y_3')^2}{2\gamma}\right\}$$

$$\left[H_1^{(0)}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}, \frac{Y_2-Y_2'}{\sqrt{\gamma}}\right)Y_1 + H_2^{(0)}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}, \frac{Y_2-Y_2'}{\sqrt{\gamma}}\right)Y_2 - \right.$$

$$\left. -H_1^{(0)}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}, \frac{Y_2-Y_2'}{\sqrt{\gamma}}\right)\frac{Y_1-Y_1'}{\gamma} - H_2^{(0)}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}, \frac{Y_2-Y_2'}{\sqrt{\gamma}}\right)\frac{Y_2-Y_2'}{\gamma}\right] =$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \exp\left\{-\frac{(Y_1-Y_1')^2+(Y_2-Y_2')^2+(Y_3-Y_3')^2}{2\gamma}\right\}$$

$$\left\{-\frac{\gamma-1}{\sqrt{\gamma}}\left[H_1^{(0)}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}, \frac{Y_2-Y_2'}{\sqrt{\gamma}}\right)\frac{Y_1-Y_1'}{\sqrt{\gamma}} + H_2^{(0)}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}, \frac{Y_2-Y_2'}{\sqrt{\gamma}}\right)\frac{Y_2-Y_2'}{\sqrt{\gamma}}\right] + \right.$$

$$\left. +\sqrt{1-\gamma}\left[H_1^{(0)}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}, \frac{Y_2-Y_2'}{\sqrt{\gamma}}\right)\frac{Y_1'}{\sqrt{1-\gamma}} + H_2^{(0)}\left(\frac{Y_1-Y_1'}{\sqrt{\gamma}}, \frac{Y_2-Y_2'}{\sqrt{\gamma}}\right)\frac{Y_2'}{\sqrt{1-\gamma}}\right]\right\}$$

Let us stress again that $H_j^{(0)}(Y,s)$ depend only on Y_1, Y_2 and s. With respect to Y_3 we have the usual convolution.

Step 4. Replace the projection operator by the identity operator. It is not the reduction to the Burgers system because the incompressibility condition is preserved.

Now we shall modify the first and the last terms in (37). For the first one we can write

$$\frac{(pk^{(0)})^{\frac{5}{2}}(p-1)}{((p-1)k^{(0)})^2}\int\limits_0^{((p-1)k^{(0)})^2} d\theta_2 \int\limits_{\mathbb{R}^3} \exp\left\{-\theta_2|\kappa^{(0,0)}+\frac{Y'}{\sqrt{(p-1)k^{(0)}}}|^2\right\}\cdot$$
$$\cdot\exp\left\{-s|\kappa^{(0)}+(Y-Y')\sqrt{pk^{(0)}}|^2\right\}\left\langle v(\kappa^{(0)}+(Y-Y')\sqrt{pk^{(0)}},0),\kappa^{(0,0)}+\frac{Y}{\sqrt{pk^{(0)}}}\right\rangle\cdot \tag{38}$$
$$\cdot P_{\kappa^{(0)}+\frac{Y}{\sqrt{pk^{(0)}}}}\tilde{g}_{p-1}\left(Y'\sqrt{\frac{p}{p-1}},s(1-\frac{\theta_2}{((p-1)k^{(0)})^2})\right)d^3Y'$$

The factor $(p-1)$ comes from the inductive assumption concerning h_{p-1}. As before, we replace $\exp\left\{-\theta_2|\kappa^{(0,0)}+\frac{Y'}{\sqrt{(p-1)k^{(0)}}}|^2\right\}$ by $\exp\{-\theta_2\}$, $P_{\kappa^{(0)}+\frac{Y}{\sqrt{pk^{(0)}}}}$ by the identity operator and $\tilde{g}_{p-1}\left(Y'\sqrt{\frac{p}{p-1}},s(1-\frac{\theta_2}{((p-1)k^{(0)})^2})\right)$ by $\tilde{g}_{p-1}(Y'\sqrt{\frac{p}{p-1}},s)$. All corrections are included in the remainder terms.

For the Gaussian term in $v(\kappa^{(0)}+(Y-Y')\sqrt{pk^{(0)}},0)$ we can write $\frac{1}{(2\pi)^{\frac{3}{2}}}\exp\left\{\frac{|Y-Y'|^2p}{2}\right\}$. This shows that typically $Y-Y'=O(\frac{1}{\sqrt{p}})$. For the third component $F^{(1)}$ of $v(\kappa^{(0)}+(Y-Y')\sqrt{pk^{(0)}},0)$ using the incompressibility condition we can write

$$F^{(1)}(\kappa^{(0)}+(Y-Y')\sqrt{pk^{(0)}},0)=$$
$$-\frac{1}{\sqrt{k^{(0)}}}\left((Y_1-Y_1')\sqrt{p}H_1^{(0)}((Y-Y')\sqrt{p})+(Y_2-Y_2')\sqrt{p}H_1^{(0)}((Y-Y')\sqrt{p})+O(\frac{1}{\sqrt{k^{(0)}}})\right)\cdot$$
$$\cdot\exp\left\{-\frac{p|Y-Y'|^2}{2}\right\}$$

For the inner product in (38)

$$\sqrt{pk^{(0)}}\left\langle v(\kappa^{(0)}+(Y-Y')\sqrt{pk^{(0)}},0),\kappa^{(0,0)}+\frac{Y}{\sqrt{pk^{(0)}}}\right\rangle=\exp\left\{-\frac{p|Y-Y'|^2}{2}\right\}\cdot$$
$$\cdot\Big[H_1^{(0)}((Y-Y')\sqrt{p})Y_1+H_1^{(0)}((Y-Y')\sqrt{p})Y_2-$$
$$-\sqrt{p}\left((Y_1-Y_1')\sqrt{p}H_1^{(0)}((Y-Y')\sqrt{p})+(Y_2-Y_2')\sqrt{p}H_1^{(0)}((Y-Y')\sqrt{p})\right)+O(\frac{1}{\sqrt{k^{(0)}}})\Big]$$

The expression in the square brackets grows as $\sqrt{p}$ and therefore

$$\sqrt{pk^{(0)}}\left\langle v(\kappa^{(0)}+(Y-Y')\sqrt{pk^{(0)}},0),\kappa^{(0,0)}+\frac{Y}{\sqrt{pk^{(0)}}}\right\rangle=$$

can be replaced by

$$-\sqrt{p}\left[(Y_1-Y_1')\sqrt{p}H_1^{(0)}((Y-Y')\sqrt{p})+(Y_2-Y_2')\sqrt{p}H_1^{(0)}((Y-Y')\sqrt{p})-\right.$$

$$\left.-\frac{1}{\sqrt{p}}\left(H_1^{(0)}((Y-Y')\sqrt{p})Y_1+H_1^{(0)}((Y-Y')\sqrt{p})Y_2\right)\right]$$

Further,

$$\exp\{-s|\kappa^{(0)}+(Y-Y')\sqrt{pk^{(0)}}|^2\}=\exp\{-s|k^{(0)}|^2\}\cdot$$

$$\cdot\exp\{-2sk^{(0)}\langle\kappa^{(0,0)},(Y-Y')\sqrt{p}\sqrt{k^{(0)}}\rangle\}\exp\{-s|Y-Y'|^2pk^{(0)}\}$$

The first factor takes values $O(1)$, the others can be written as $1+O(\frac{1}{\sqrt{k^{(0)}}})$. The main order of magnitude of (38) takes the form

$$p\exp\{-s(k^{(0)})^2\}\frac{(p-1)}{p}\frac{1}{p}\left[-\int_{\mathbb{R}^3}\left[(Y_1-Y_1')\sqrt{p}H_1^{(0)}((Y-Y')\sqrt{p})+(Y_2-Y_2')\sqrt{p}H_1^{(0)}((Y-Y')\sqrt{p})\right]+\right.$$

$$+\frac{1}{\sqrt{p}}\left[H_1^{(0)}((Y-Y')\sqrt{p})Y_1+H_1^{(0)}((Y-Y')\sqrt{p})Y_2\right]\left(\frac{p}{2\pi}\right)^{\frac{3}{2}}\exp\left\{-\frac{|Y-Y'|^2p}{2}\right\}\cdot$$

$$\left.\cdot\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\exp\left\{-\frac{|Y'|^2p}{2(p-1)}\right\}H^{(0)}\left(Y'\sqrt{\frac{p}{p-1}}\right)d^3Y'\right]$$

A similar expression can be written for the last term in (37). Remark that due to our choice of the interval $S^{(1)}$ the product $s(k^{(0)})^2=O(1)$.

Now we derive the recurrent formula for $Z_p(s)$ and $\Lambda_p(s)$. Since our special fixed point $H^{(0)}$ is a Hermite polynomial of first degree, the convolution of $H^{(0)}$ over Y' (see (37)) gives us simply the product of $H^{(0)}$ and the Gaussian term and a polynomial in γ. The function $H^{(0)}$ and the Gaussian term can then be taken out of the summation in γ and this gives us the following recurrent system for $Z_p(s)$ and $\Lambda_p(s)$:

$$Z_{p+1}(s)\Lambda_{p+1}^p(s)$$
$$=\sum_{p_1+p_2=p}\frac{1}{p}\cdot\frac{i}{(k^{(0)})^2}\cdot(6\gamma^2-10\gamma+4)\cdot Z_p(s)^2\cdot\Lambda_p^p(s)\cdot(1-e^{-s(p_1k^{(0)})^2})\cdot(1-e^{-s(p_2k^{(0)})^2})\tag{39}$$

where the factor $(6\gamma^2 - 10\gamma + 4)$ comes from the convolution of $H^{(0)}$ with itself. Now if we take $Z_p(s) = -i(k^{(0)})^2$ and write $\frac{\Lambda_{p+1}(s)}{\Lambda_p(s)} = 1 + \frac{\xi_{p+1}}{p^2}$, then we have

$$\left(1 + \frac{\xi_{p+1}}{p^2}\right)^p = \sum_\gamma \frac{1}{p} \cdot (6\gamma^2 - 10\gamma + 4) \cdot (1 - e^{-s(p_1 k^{(0)})^2})(1 - e^{-s(p_2 k^{(0)})^2})$$

then it is not difficult to see that there exists bounded ξ_{p+1} (with an bound independent of p) such that the equality holds. It is an elementary fact that the limit

$$\Lambda(s) = \lim_{p \to \infty} \Lambda_{p+1}(s) = \Lambda_1 \prod_{k=1}^{\infty} \left(1 + \frac{\xi_{k+1}}{k^2}\right)^k$$

exists.

Now we discuss the behavior of all remainders for $p < \left(k^{(0)}\right)^{\lambda_2}$.

By $\Phi_j^{(u)}$, $\Phi_{j'}^{(n)}$ we denote the eigen-vectors of the linearized renormalization group corresponding to the fixed point $H^{(0)}$. For each p we make the following inductive assumption for $\delta^{(r)}(Y,s)$, $r < p$:

$$\delta^{(r)}(Y,s) = \sum_{j=1}^{4} \left(b_{j,r}^{(u)} + \beta_{j,r}^{(u)}\right) \Phi_j^{(u)} + \sum_{j'=1}^{6} \left(b_{j',r}^{(n)} + \beta_{j',r}^{(n)}\right) \Phi_{j'}^{(n)} + \Phi_r^{(st)}, \quad \gamma = \frac{r}{p-1}$$

where $b_{j,r}^{(u)} = (p-1)^{\alpha_j} b_j^{(u)} \gamma^{\alpha_j^{(u)}}$, $b_{j',r}^{(n)} = b_{j'}^{(n)}$, $\Phi_r^{(st)}$ is a function which belongs to the stable subspace of the linearized renormalization group, $\gamma = \frac{r}{p-1}$.

As we go from $p-1$ to p, the variable $\gamma = \frac{r}{p-1}$ is replaced by $\gamma' = \frac{r}{p} = \gamma \cdot \frac{p-1}{p}$. Therefore

$$\begin{aligned}\left(b_{j,r}^{(u)} + \beta_{j,r}^{(u)}\right) \gamma^{\alpha_j} \Phi_j^{(u)} &= \left((p-1)^{\alpha_j} b_j^{(u)} + \beta_{j,r}^{(u)}\right) \cdot \left(\frac{p}{p-1}\right)^{\alpha_j} \cdot (\gamma')^{\alpha_j} \Phi_j^{(u)} \\ &= \left(p^{\alpha_j} b_j^{(u)} + \left(\frac{p}{p-1}\right)^{\alpha_j} \cdot \beta_{j,r}^{(u)}\right) \cdot (\gamma')^{\alpha_j} \Phi_j^{(u)}.\end{aligned}$$

In the same way for the neutral eigen-functions we have

$$\left(b_{j'}^{(n)} + \beta_{j',r}^{(n)}\right) \Phi_{j'}^{(n)}$$

because $\alpha_{j'} = 0$. In the same way one can transform $\Phi^{(st)}$.The coefficients $\beta_{j,r}^{(u)}$, $\beta_{j',r}^{(n)}$ are small compared to the first term . An important conclusion is that the projections to the unstable

directions increase, projections to the neutral directions remain the same and projections to the stable directions decrease. As was already said, in the case of unstable and neutral directions the term containing $b_j^{(u)}$ or $b_{j'}^{(n)}$ is the main term.

Now we discuss the form of $\delta^{(p)}(Y,s)$. It is the sum of three types of terms.

a_1). The term which depends linearly on all $\delta^{(r)}(Y,s)$. Especially important is the part which contains all $p^{\alpha_j} b_j^{(u)} (\gamma')^{\alpha_j} \Phi_j^{(u)}$, $b_{j'}^{(n)} \Phi_{j'}^{(n)}$. If we were to have and be in the limiting regime $H^{(0)}$ then the integral will give $p^{\alpha_j} b_j^{(u)} \left(1+\frac{1}{p}\right)^{\alpha_j} \Phi_j^{(u)} = (p+1)^{\alpha_j} b_j^{(u)} \Phi_j^{(u)}$ since $\gamma' = 1$. However, $H^{(r)}$ are slightly different from $H^{(0)}$. Therefor we shall have a small correction which is included in all $\beta_{j,p}^{(u)}$, $\beta_{j',p}^{(n)}$ and in $\Phi_p^{(st)}$. We denote it as $\beta_{pj1}^{(u)}$, $\beta_{pj'1}^{(n)}$, $\Phi_{p,1}^{(st)}$. The we have terms which are linear functions of all $\beta_{j,r}^{(u)}$, $\beta_{j',r}^{(n)}$ and $\Phi_r^{(st)}$. They will give us $\beta_{pj2}^{(u)}$, $\beta_{pj'2}^{(n)}$, $\Phi_{p,2}^{(st)}$.

a_2). The term which is the sum of all quadratic expressions depending on $\delta^{(p_1)}$, $\delta^{(p_2)}$. We expand it using our basis of $\Phi_j^{(u)}$, $\Phi_{j'}^{(n)}$ and all stable eigen-vectors.

a_3). The term which contains all corrections which arise during the four steps described above. We also expand it in the same way as in a_2)).

The sum of all terms gives $\beta_{p,j}^{(u)}$, $\beta_{p,j'}^{(n)}$, Φ_p^{st}.

We use this procedure till $p = p_0 = (k^{(0)})^{\lambda_2}$. The procedure for $p > p_0$ is discussed in §9.

§8. The List of Remainders and Their Estimates

In the beginning of §7 we described 10-parameter families of initial conditions which we consider in this paper. We mentioned above that for each p we have an interval $S^{(p)} = \left[S_-^{(p)}, S_+^{(p)}\right]$ on the time axis. Actually these intervals are changing when $p = p_n = (1+\epsilon)^n$ where $\epsilon > 0$ is a constant. Therefore we shall write $S^{(n)} = \left[S_-^{(p_n)}, S_+^{(p_n)}\right]$ and hope that there will be no confusion.

In this and the next section we consider $p > (k^{(0)})^{\lambda_2}$. Each function $\tilde{g}_r(Y,s)$, $3 \le r < p$, has the following representation:

in the domain $|Y| \leq C_1 \sqrt{\ln r k^{(0)}}$, $Y = (Y_1, Y_2, Y_3) \in \mathcal{R}^3$

$$\tilde{g}_r(Y, s) \ = \ \Lambda^{r-1} \ \cdot \ r \ \cdot \ \frac{\sigma^{(1)}}{2\pi} \ \exp\left\{ \frac{\sigma^{(1)}}{2} \ \left(|Y_1|^2 + \right.\right.$$

$$\left.\left. + \, |Y_2|^2 \right) \right\} \ \cdot \ \sqrt{\frac{\sigma^{(2)}}{2\pi}} \ \exp\left\{ -\frac{\sigma^{(2)}}{2} \ |Y_3|^2 \right\} \ \cdot \ \left(H^{(0)}(Y_1, Y_2) \ + \ \delta^{(r)}(Y, s) \right);$$

in the domain $|Y| > C_1 \sqrt{\ln(r k^{(0)})}$:

$$\begin{aligned} & \frac{\sigma^{(1)}}{2\pi} \ \exp\left\{ - \ \frac{\sigma^{(1)}}{2} \ (|Y_1|^2 \ + \ |Y_2|^2) \right\} \ \cdot \ \sqrt{\frac{\sigma^{(2)}}{2\pi}} \cdot \\ & \quad \cdot \exp\left\{ - \ \frac{\sigma^{(2)}}{2} \ |Y_3|^2 \right\} \ \cdot \ |H^{(0)} \ (Y_1, Y_2) \ + \ \delta^{(r)}(Y, s)| \\ & \leq \ \Lambda^{r-1} \cdot r \cdot \frac{1}{r^{\lambda_3 - 1}} \end{aligned}$$

for some constant $\lambda_3 > 0$. We use the formula (7) to get $\tilde{g}^{(p)}(Y, s)$. New remainders appear in one of the following ways.

Type I. The recurrent relation (7) does not coincide with the equation for the fixed point and actually is some perturbation of this equation. The difference produces some remainders which tend to zero as $p \to \infty$.

Type II. For the limiting equation all eigen-vectors in the linear approximation are multiplied by some constant. In the equation (7) it is no longer true and the difference generate some remainders. (see also §9).

Type III. The remainders which are quadratic functions of all previous remainders.

§8A. The Remainders of Type I.

We call the domain A the set $\{|Y| \leq D_1\sqrt{\ln(rk^{(0)})}\}$ and the domain B the set $\{|Y| > D_1\sqrt{\ln(rk^{(0)})}\}$. The estimates will be done separately in each domain. We include the first, the second and the last two terms in (7) in the remainders. We shall estimate only the first one, the others are estimated in the same way.

Domain A: We have

$$\beta_p^{(1)}(Y,s) = (p+1)^{\frac{5}{2}} \cdot \frac{i}{sp^2} \cdot \int_0^{p^2} d\theta_2 \int_{\mathbb{R}^3} < v\left(\left(k^{(0)} + \frac{Y-Y'}{\sqrt{s}}\sqrt{p+1}\,,\,0\right)\,;\,b\right),$$

$$\sqrt{s}\,k^{(0)} + \frac{Y}{\sqrt{p+1}} > P_{\sqrt{s}\,k^{(0)}+\frac{Y}{\sqrt{p+1}}}\,\tilde{g}_p\left(Y', \left(1-\frac{\theta_2}{p^2}\right)s\right)$$

$$\exp\left\{-\left|\sqrt{s}\,k^{(0)} + (Y-Y')\sqrt{p+1}\right|^2 - \frac{\theta_2}{p^2}\left|\sqrt{s}\,k^{(0)}\,p \,+\, Y'\sqrt{p+1}\right|^2\right\} d^3Y'$$

Here b means the collection of all parameters of $v(k;0)$. The the main contribution to the integral comes from $Y - Y' = O\left(\frac{1}{\sqrt{p+1}}\right)$. In this domain in the main order of magnitude

$$\langle v(k^{(0)} + \frac{Y-Y'}{\sqrt{s}}\sqrt{p+1},\,0\,;\,b),\,\sqrt{s}\,k^{(0)}\rangle = O(1)$$

Assuming that $v(k^{(0)} + \frac{Y-Y'}{\sqrt{s}}\sqrt{p+1},\ 0;\ b)$ is differentiable `w.r.t` the first three variables we see that the inner product

$$\langle v(k^{(0)} + \frac{Y-Y'}{\sqrt{s}}\sqrt{p+1},\,0\,;\,\alpha)\,,\sqrt{s}\,k^{(0)} + \frac{Y}{\sqrt{p+1}}\rangle$$

is of order $O(1)$. For $\tilde{g}_p$ we can write using our inductive assumptions

$$\tilde{g}_p\left(Y', \left(1-\frac{\theta_2}{p^2}\right)s\right) = \Lambda^{p-1}\cdot p \cdot \frac{\sigma^{(1)}}{2\pi}\cdot\sqrt{\frac{\sigma^{(2)}}{2\pi}}\cdot\exp\left\{-\frac{\sigma^{(1)}\left(|Y_1|^2 + Y_2|^2\right)}{2}\right\}$$

$$\cdot\exp\left\{-\frac{\sigma^{(2)}\left(|Y_3|^2\right)}{2}\right\}\cdot\mathcal{H}^{(p)}\left(Y', \left(1-\frac{\theta_2}{p^2}\right)s\right).$$

Also

$$\exp\left\{-\frac{\theta_2}{p^2}\left|\sqrt{s}\,k^{(0)}\,p\,+\,Y'\,\sqrt{p+1}\right|^2\right\} = \exp\left\{-\theta_2\left|\sqrt{s}\,k^{(0)}\,+\,\frac{Y'\sqrt{p+1}}{p}\right|^2\right\}$$

and in the main order of magnitude the integration over θ_2 does not depend on Y'. Thus we can write

$$|\beta_p^{(1)}(Y,s)| \,\leq\, \Lambda^{(p-2)}\,\cdot\,p\,\cdot\,\exp\left\{-\frac{\sigma^{(1)}}{2}\,(|Y_1|^2\,+\,|Y_2|^2)\right\}$$

$$\cdot\,\exp\left\{-\frac{\sigma^{(2)}}{2}\,|Y_3|^2\right\}\,\cdot\,\frac{D_4}{p} \tag{40}$$

Here and later various constants whose exact values play no role in the arguments will be denoted by the letter D with indices. Since in the expression for $\tilde{g}_{p+1}$ we have the factors $\Lambda^p\,\cdot\,(p+1)\,\cdot\,\exp\left\{-\frac{\sigma^{(1)}}{2}\,(|Y_1|^2\,+\,|Y_2|^2\right\}\,\cdot\,\frac{\sigma^{(1)}}{2\pi}\,\sqrt{\frac{\sigma^{(2)}(s)}{2\pi}}\,\cdot\,\exp\left\{-\frac{\sigma^{(2)}}{2}\,|Y_3|^2\right\}$, the estimate (40) shows that $|\beta_p^{(1)}(Y,s)|$ is relatively smaller than $\tilde{g}_{p+1}$ with an order $O(\frac{1}{p})$. This is good enough for our purposes. We did not discuss the errors which follow from the fact that the expressions in the previous formulas depend on θ_2.

<u>Domain B</u>: The smallness of $\beta_p^{(1)}(Y,s)$ in this case follows easily from several inequalities and arguments.

1°: $|Y|\,\leq\,D_4\,\sqrt{pk^{(0)}}$ because $|k|\,\leq\,D_5pk^{(0)}$.

2°: $|Y-Y'|\,\leq\,D_6\sqrt{k^{(0)}}$ because $v(k,0;b)$ has a compact support.

3°: If $|Y-Y'\,|\,\leq\,\frac{2s_+}{\sqrt{p}}$ then

$$\exp\left\{-\,\left|\sqrt{s}\,k^{(0)}+(Y-Y')\,\sqrt{p+1}\right|^2\right\}\leq 1$$

If $|Y-Y'|\,\geq\,\frac{2s_+}{\sqrt{p}}$ then

$$\exp\left\{-\left|\sqrt{s}\,k^{(0)}\,+\,(Y-Y')\sqrt{p+1}\right|^2\right\}\,\leq\,\exp\left\{-\frac{s_+}{4}\left|Y-Y'\right|^2\right\}.$$

4°: If $|Y'| \geq D_7 \sqrt{p}$ then

$$\exp\left\{-\frac{\theta_2}{p^2}\left|\sqrt{s}\,k^{(0)}p + Y'\sqrt{p+1}\right|^2\right\} \leq \exp\{-C_8\theta_2\}$$

5°: If $|Y'| \leq D_7 \sqrt{p}$ then

$$\exp\left\{-\frac{\theta_2}{p^2}\left|\sqrt{s}\,k^{(0)}\,p + Y'\,\sqrt{p+1}\right|^2\right\} \leq 1\,.$$

6°: We have

$$\exp\left\{-\frac{\sigma^{(1)}}{2}\left(|Y_1'|^2 + |Y_2'|^2\right) - \frac{\sigma^{(2)}}{2}|Y_3'|^2\right\}$$

$$= \exp\left\{-\frac{\sigma^{(1)}}{2}\Big(|Y_1 - (Y_1 - Y_1')|^2 + |Y_2 - (Y_2 - Y_2')|^2\Big)\right.$$

$$\left. - \frac{\sigma^{(2)}}{2}|Y_3 - (Y_3 - Y_3')|^2\right\}$$

$$= \exp\left\{-\frac{\sigma^{(1)}}{2}\left(|Y_1|^2 + |Y_2|^2\right) - \frac{\sigma^{(2)}}{2}\left(|Y_3|^2\right)\right\}$$

$$\cdot \exp\left\{\sigma^{(1)}\left(Y_1(Y_1 - Y_1') + Y_2(Y_2 - Y_2')\right) + \sigma^{(2)}\,Y_3(Y_3 - Y_3')\right.$$

$$\left. - \frac{\sigma^{(1)}(s)}{2}\Big(|Y_1 - Y_1'|^2 + |Y_2 - Y_2'|^2\Big) - \frac{\sigma^{(2)}}{2}|Y_3 - Y_3'|^2\right\}.$$

If $|Y - Y'| \leq \dfrac{2s_+}{\sqrt{p}}$ then

$$\exp\left\{\sigma^{(1)}\left(Y_1(Y_1 - Y_1') + Y_2(Y_2 - Y_2')\right) + \sigma^{(2)}(s)\,Y_3(Y_3 - Y_3')\right.$$

$$\left. - \frac{\sigma^{(1)}}{2}\left(|Y_1 - Y_1'|^2 + |Y_2 - Y_2'|^2\right) - \frac{\sigma^{(2)}}{2}|Y_3 - Y_3'|^2\right\} \leq C_8\,.$$

If $|Y - Y'| > \dfrac{2s_+}{\sqrt{p}}$ then we have an integral of the function which is the product of some Gaussian factor and $|\mathcal{H}^{(p)}(Y)|$. Direct estimate shows as before that in this case

$$|\beta_p^{(1)}(Y,s)| \le \Lambda^{(p-1)} \cdot p \cdot e^{-\dfrac{\sigma^{(1)}}{2}(|Y_1|^2 + Y_2|^2)} \cdot e^{-\dfrac{\sigma^{(2)}}{2}|Y_3|^2} \cdot \frac{D_8}{p^{\frac{3}{2}}}$$

which is also good for us.

In the same way one can estimate terms with relatively small p_1 and $p - p_1$ (i.e., $p_1 \le \sqrt{p}$ or $p_1 \ge p - \sqrt{p}$. The remainders will be of order $\frac{1}{\sqrt{p_1}} \cdot \frac{1}{p}$. The next set of remainders comes from splitting the integration over θ and Y' (see (7) and beginning of §3). We may assume that $p_1 > \sqrt{p}$ or $p_1 < p - \sqrt{p}$ because other terms were estimated before. Put

$$\tilde{\tilde{g}}_{p+1}(Y,s) = i\,(p+1)^{\frac{5}{2}} \sum_{\substack{p_1+p_2=p+1\\ p_1,p_2>\sqrt{p}}} \int_0^{p_1^2} d\theta_1 \int_0^{p_2^2} d\theta_2 \cdot \frac{1}{p_1^2 p_2^2}$$

$$\int_{\mathbb{R}^3} \langle \tilde{g}_{p_1}\left((Y-Y')\,\frac{(1-\frac{\theta_1}{p_1^2})^{\frac{1}{2}}}{\sqrt{\gamma}}, \left(1 - \frac{\theta_1}{p_1^2}\right) s\right), \sqrt{s}\,k^{(0)} + \frac{Y}{\sqrt{p+1}}\rangle$$

$$P_{\sqrt{s}\,k^{(0)} + \dfrac{Y}{\sqrt{p+1}}}\ \tilde{g}_{p_2}\left(\frac{Y'(1-\frac{\theta_2}{p_2^2})^{\frac{1}{2}}}{\sqrt{(1-\gamma)}}, \left(1 - \frac{\theta_2}{p_2^2}\right) s\right)$$

$$\cdot \exp\left\{ -\theta_1 |\sqrt{s}\,k^{(0)} + \frac{Y - Y'}{\sqrt{p+1}\cdot\gamma}|^2 - \theta_2 |\sqrt{s}\,k^{(0)} + \frac{Y - Y'}{\sqrt{p+1}\gamma}|^2 \right\}.$$

Using the inductive assumption we can rewrite the last expression as follows:

$$\tilde{\tilde{g}}_{p+1}(Y,s) = i\,(p+1) \sum_{\substack{p_1+p_2=p+1\\ p_1,p_2>\sqrt{p}}} \int_0^{p_1^2} d\theta_1 \int_0^{p_2^2} d\theta_2$$

$$\Lambda^{p_1-1} \cdot \Lambda^{p_2-1} \cdot \frac{1}{\gamma(1-\gamma)} \cdot \frac{1}{p+1} \exp\left\{ -\frac{\sigma^{(1)}}{2} \frac{|Y_1 - Y_1'|^2 + |Y_2 - Y_2'|^2}{\gamma} \right.$$

$$- \frac{\sigma^{(2)}}{2} \frac{|Y_3 - Y_3'|^2}{\gamma} - \frac{\sigma^{(1)}}{2} \frac{|Y_1'|^2 + |Y_2'|^2}{(1-\gamma)}$$

$$- \frac{\sigma^{(1)}}{2} \cdot \frac{|Y_3'|^2}{1-\gamma} \Big\} . p^{\frac{1}{2}} < \mathcal{H}^{(p_1)} \left(Y - Y',\, s \left(1 - \frac{\theta_1}{p_1^2} \right) \right) ,$$

$$\sqrt{s}\, k^{(0)} + \frac{Y}{\sqrt{p}} > \cdot P_{\sqrt{s}\, k^{(0)} + \frac{Y}{\sqrt{p}}} \mathcal{H}^{(p_2)} \left(Y', s \left(1 - \frac{\theta_2}{p_2^2} \right) \right) d^3Y' .$$

As was explained before, due to incompressibility in the Domain A , the inner product

$$\langle \mathcal{H}^{(p_1)} \left(Y - Y';\, s \left(1 - \frac{\theta_1}{p_1^2} \right) \right) , \sqrt{s}\, k^{(0)} + \frac{Y}{\sqrt{p}} \rangle$$

takes values $O(\frac{1}{\sqrt{p}})$ because the first two components of the vector $\sqrt{s}\, k^{(0)} + \frac{Y}{\sqrt{p}}$ are of order $O(\frac{1}{\sqrt{p}})$. Therefore the product

$$\sqrt{p}\, \langle\, \mathcal{H}^{(p_1)} \left(Y - Y',\, s \left(1 - \frac{\theta_1}{p_1^2} \right) \right) , \sqrt{s}\, k^{(0)} + \frac{Y}{\sqrt{p}} \rangle$$

takes values of order $O(1)$.

The remainder can be written in the following form:

$$\beta_p^{(2)} (Y, s) = i \sum_{\substack{p_1 + p_2 = p+1 \\ p_1, p_2 > \sqrt{p}}} \frac{1}{\gamma(1-\gamma)} \cdot \frac{1}{p} \cdot \int_0^{p_1^2} d\theta_1 \int_0^{p_2^2} d\theta_2$$

$$\cdot \Lambda^{p_1 - 1} \cdot \Lambda^{p_2 - 1} \cdot \frac{1}{\Lambda^p} \cdot \int_{\mathbb{R}^3} \exp \Big\{ - \frac{\sigma^{(1)} \left(|Y_1 - Y_1'|^2 + |Y_2 - Y_2'|^2 \right)}{2\gamma}$$

$$- \frac{\sigma^{(2)}}{2\gamma} \cdot \frac{|Y_3 - Y_3'|^2}{2\gamma} - \frac{\sigma^{(1)} \left(|Y_1'|^2 + Y_2'|^2 \right)}{2(1-\gamma)} - \frac{\sigma^{(2)} |Y_3'|^2}{2(1-\gamma)} \Big\} < \mathcal{H}^{(p_1)} \left(Y - Y',\, s \left(1 - \frac{\theta_1}{p_1^2} \right) \right) ,$$

$$\sqrt{s}\,k^{(0)} \;+\; \frac{Y}{\sqrt{p}} \;>\; \cdot P_{\sqrt{s}\,k^{(0)}+\frac{Y}{\sqrt{p}}}\,\mathcal{H}^{(p_2)}\,\left(Y',\left(1-\frac{\theta_2}{p_2^2}\right)s\right)\cdot$$

$$\cdot\,\exp\left\{-\theta_1|\sqrt{s}\,k^{(0)}\;+\;\frac{Y-Y'}{\sqrt{p}\gamma}|^2\;-\;\theta_2|\sqrt{s}\,k^{(0)}\;+\;\frac{Y'}{\sqrt{p}\,(1-\gamma)}|^2\right\}\cdot$$

$$-\,i\sum_{\substack{p_1+p_2=p+1\\ p_1,p_2>1}}\frac{1}{\gamma(1-\gamma)}\cdot\frac{1}{p}\cdot\int_0^{p_1^2}\exp\left\{-\theta_1 s\right\}\,d\theta_1\int_0^{p_2^2}\exp\left\{-\theta_2 s\right\}\,d\theta_2$$

$$\int_{\mathbb{R}^3}\exp\left\{-\,\frac{\sigma^{(1)}\,(|Y_1-Y_1'|^2\;+\;|Y_2-Y_2'|^2)}{2\gamma}\;-\;\frac{\sigma^{(2)}(|Y_3-Y_3'|^2)}{2\gamma}\right.$$

$$\left.-\,\frac{\sigma^{(1)}\,(|Y_1'|^2\;+\;Y_2'|^2)}{2(1-\gamma)}\;-\;\frac{\sigma^{(2)}\,|Y_3'|^2}{2(1-\gamma)}\right\}$$

$$\cdot p^{\frac{1}{2}}\;\cdot\langle\mathcal{H}^{(p_1)}\,(Y-Y'),\,\sqrt{s}\,k^{(0)}\;+\;\frac{Y}{\sqrt{p}}\rangle\,P_{\sqrt{s}\,k^{(0)}\;+\;\frac{Y}{\sqrt{p}}}\,\mathcal{H}^{(p_2)}\,(Y',s)\,d^3Y'\,.$$

We did not include the factor $\Lambda^{p-1}\cdot p$ because it is a part of the inductive assumption. This remainder is estimated in the following way.

First we consider

$$R_1\;=\;\left(\left|\sqrt{s}\,k^{(0)}\;+\;\frac{Y-Y'}{\sqrt{p}\gamma}\right|^2-s\right)\;+\;\left(\left|\sqrt{s}\,k^{(0)}\;+\;\frac{Y'}{\sqrt{p}\,(1-\gamma)}\right|^2-s\right)$$

As before, consider the domain where

$$|Y-Y'|\;\leq\;D_9\,\sqrt{\ln{(pk^{(0)})}},\;|Y'|\;\leq\;D_{10}\,\sqrt{\ln{(pk^{(0)})}}\,.$$

We write

$$R_1 = \frac{|Y-Y'|^2}{p \cdot \gamma_1^2} + \frac{|Y'|^2}{p \ \cdot \gamma_2^2} + C_{11}\left(\frac{|Y-Y'|}{\sqrt{p}\,\gamma|} + \frac{|Y'|}{\sqrt{p}(1-\gamma)}\right).$$

In the Domain A

$$|R_1| \le \frac{C_{12}\ln(pk^{(0)})}{pk^{(0)}}.$$

Therefore

$$R_2 = \exp\left\{-\theta_1\left|\sqrt{s}\,k^{(0)} + \frac{Y-Y'}{\sqrt{p}\,\gamma}\right|^2 - \theta_2\left|\sqrt{s}\,k^{(0)} + \frac{Y'}{\sqrt{p}\gamma^2}\right|^2\right\}$$
$$- \exp\{-\theta_1 s\} \cdot \exp\{-\theta_2 s\}$$

$$= \exp\{-(\theta_1+\theta_2)s)\} \cdot \left[\exp\left\{-\theta_1\left(\left|\sqrt{s}\,k^{(0)} + \frac{Y-Y'}{\sqrt{p}\gamma}\right|^2 - s\right)\right.\right.$$
$$\left.\left.\cdot \exp\left\{-\theta_2\left(\left|\sqrt{s}\,k^{(0)} + \frac{Y'}{\sqrt{p}(1-\gamma)}\right|^2 - s\right)\right\} - 1\right]$$

and in the Domain A

$$|R_2| \le \exp\{-(\theta_1 + \theta_2)s\}\left(\frac{\theta_1 \cdot C_{13}}{\sqrt{p}\gamma} + \frac{\theta_2 \ln p}{\sqrt{p}(1-\gamma)}\right).$$

This shows that in the Domain A we can replace the exponent

$$\exp\left\{-\theta_1|\sqrt{s}\,k^{(0)} + \frac{Y-Y'}{\sqrt{p}\gamma}|^2 - \theta_2|\sqrt{s}\,k^{(0)} + \frac{Y'}{\sqrt{p}(1-\gamma)}|^2\right\}$$

by $\exp\{-(\theta_1+\theta_2)s(k^{(0)})^2\}$ and the remainder will be not more than $\frac{D_{14}\ln p}{\sqrt{p}}$. This is enough for our purposes. In the Domain B the estimates are similar because again the main contribution to the integral comes from $|Y-Y'| \le D_9\sqrt{\ln p}$, $|Y'| \le D_{10}\sqrt{\ln p}$. In other words, in the Domain B we can replace the product of the Gaussian factors and $\mathcal{H}^{(p)}$ by

$$\exp\left\{-\frac{1}{2}\sigma^{(1)}\left(|Y_1-Y_1'|^2 + |Y_2-Y_2'|^2\right) - \frac{1}{2}\sigma^{(2)}|Y_3-Y_3'|^2\right.$$

$$- \frac{1}{2}\,\sigma^{(1)}\,(|Y_1'|^2 \;+\; |Y_2'|^2) \;-\; \frac{1}{2}\,\sigma^{(2)}(|Y_3'|^2) \Big\} \;.$$

This is also enough for our purpose.

The next remainder of Type I comes from the difference between the sum over γ and the corresponding integral. The remainder $\beta_p^{(3)}(Y,s)$ is the difference between the sum

$$i \sum_{\substack{p_1+p_2=p+1 \\ p_1,p_2>\sqrt{p}}} \sqrt{\gamma}\,\sqrt{(1-\gamma)} \cdot \frac{1}{p} \cdot \int\limits_{\mathbb{R}^3} \exp\Big\{ - \frac{\sigma^{(1)}\,(|Y_1-Y_1'|^2 \;+\; |Y_2-Y_2'|^2)}{2\gamma}$$

$$- \frac{\sigma^{(2)}(|Y_3-Y_3'|)^2}{2\gamma} - \frac{\sigma^{(1)}(|Y_1'|^2 \;+\; |Y_2'|^2)}{2(1-\gamma)} - \frac{\sigma^{(2)}|Y_3'|^2}{2(1-\gamma)} \Big\}$$

$$\cdot \left(\frac{1}{2\pi\gamma}\right)^{\frac{3}{2}} \cdot \left(\frac{1}{2\pi(1-\gamma)}\right)^{\frac{3}{2}} \cdot p^{\frac{1}{2}} \cdot \langle \mathcal{H}^{(p_1)}((Y-Y'))\,, \sqrt{s}\,k^{(0)} \;+\; \frac{Y}{\sqrt{p}} \rangle$$

$$P_{\sqrt{s}\,k^{(0)} \;+\; \frac{Y}{\sqrt{p}}} \, \mathcal{H}^{(p_2)}(Y',s)\, d^3Y'$$

and the corresponding integral over γ from 0 to 1. It is easy to check that this difference is not more than $\frac{C_{14}}{\sqrt{p}}$.

§8B. The Remainders of Type II and III

All remainders of Type II appear because we use the sums (over p_1) instead of the integrals . The functions $\mathcal{H}\left(\frac{Y-Y'}{\sqrt{\gamma}}\right)$ are defined for all γ. We use a linear interpolation to define $\delta(\gamma,Y,s)$ for all γ. From our inductive assumptions it follows that $|\delta_p(\gamma,Y,s)| \leq \frac{C_{16}}{\sqrt{p}}$. Therefore, the remainders which follow from the difference between the sum and the integral also satisfy this estimate.

It remains to consider quadratic expressions of $\delta_p(\gamma,Y,s)$. The Gaussian density is present in all these expressions. Therefore, all the remainders are not more than $\frac{C_{17}}{p}$.

§9. Final Steps in the Proof of the Main Result

In this section we consider our procedure for $p > p_0$. Introduce the sequence p_n, $p_n = (1+\epsilon)p_{n-1} = (1+\epsilon)^n p_0$, where $\epsilon > 0$ is small (see below). These are the values of p when we make the renormalization of our parameters. For $p \neq p_n$, no renormalization is done.

In §7 we explained the choice of our fixed point $H^{(0)}$. The corresponding eigen-functions are denoted by $\Phi_j^{(u)}$ and $\Phi_{j'}^{(n)}$. Also we have eigen-functions of the stable part of the spectrum. Consider p, $p_m < p < p_{m+1}$. By induction we assume that we have an interval on the time axis $\left[S_-^{(m)}, S_+^{(m)}\right]$ and $s \in \left[S_-^{(m)}, S_+^{(m)}\right]$, $r < p$, so that

$$\begin{aligned}\tilde{g}_r(Y,s) =& \Lambda^{r-1} \cdot r \cdot (H^{(0)}(Y) + \delta^{(r)}(Y,s)) \cdot \\ & \cdot \frac{\sigma^{(1)}}{2\pi} \exp\left\{-\frac{\sigma^{(1)}(Y_1^2+Y_2^2)}{2}\right\} \cdot \sqrt{\frac{\sigma^{(2)}}{2\pi}} \exp\left\{-\frac{\sigma^{(2)}Y_3^2}{2}\right\}\end{aligned}$$

If $\gamma = \frac{r}{p-1}$ then

$$\delta^{(r)}(Y,s) = \sum_{j=1}^{4}\left(b_{j,p}^{(u)} + \beta_{j,r}^{(u)}\right)\gamma^{\alpha_j^{(u)}}\Phi_j^{(u)}(Y) + \sum_{j'=1}^{6}\left(b_{j',p}^{(n)} + \beta_{j',r}^{(n)}\right)\Phi_j^{(n)}(Y) + \Phi_r^{(st)}(Y,\gamma).$$

here $\beta_{j,r}^{(u)}, \beta_{j',r}^{(n)}$ are small corrections to the main terms $b_{j,p}^{(u)}$, $b_{j',p}^{(n)}$, $\Phi_r^{(st)}$ can be written as a series w.r.t. the stable eigen-functions. (see Appendix II).

At one step of our procedure $p-1$ is replaced by p, γ is replaced by $\gamma' = \gamma \cdot \frac{p-1}{p}$ and $\gamma^{\alpha_j^{(u)}}$ is replaced by $\left(1+\frac{1}{p-1}\right)^{\alpha_j^{(u)}} \cdot (\gamma')^{\alpha_j^{(u)}}$, $b_{j,p}^{(u)} + \bar{\beta}_{j,r}^{(u)}$ is replaced by $(\bar{b}_{j,p}^{(u)} + \beta_{j,r}^{(u)})\left(1+\frac{1}{p-1}\right)^{\alpha_j^{(u)}}$. During the whole interval $p_m < p < p_{m+1}$ the variable $b_{j,p_m}^{(u)}$ acquires the factor

$$\prod_{p_m<p<p_{m+1}}\left(1+\frac{1}{p-1}\right)^{\alpha_j^{(u)}} \approx e^{(1+\epsilon)\alpha_j^{(u)}}.$$

A similar statement holds for the stable part of the spectrum. The neutral part remains the same since $\alpha_{j'}^{(n)} = 0$.

Now we shall discuss $\delta^{(p)}(Y,s)$ using (7). As in §7 $\delta^{(p)}(Y,s)$ consists of three parts.

Part I. In all $\delta^{(r)}$ the main term is the one which contains our basic parameters $b_j^{(u)}$, $b_{j'}^{(n)}$. We consider terms in (7) which are linear in $b_j^{(u)}$, $b_{j'}^{(n)}$. As it follows from the definition

of the linearized group and its spectrum we get $\left(1+\frac{1}{p}\right)^{\alpha_j^{(u)}} b_{j,p}^{(u)}$. For the neutral part we get 1 because $\alpha_{j'}^{(n)} = 0$. We put $b_{j,p+1}^{(u)} = b_{j,p}^{(u)} \cdot \left(1+\frac{1}{p}\right)^{\alpha_j^{(u)}} b_{j,p}^{(u)}$, $b_{j',p+1}^{(n)} = b_{j',p}^{(n)}$. The stable part is transformed accordingly.

Part II. The term which is the sum of quadratic functions of all $\delta^{(r)}$. We expand it using the basis of our functions $\Phi_j^{(u)}$, $\Phi_{j'}^{(n)}$ and the functions from the stable part of the spectrum. The result is included in $\beta_{j,p}^{(u)}$, $\beta_{j',p}^{(n)}$ and the stable function $\Phi_p^{(st)}(Y,s)$.

Part III. All remainders which arise because the formulas for finite p are different from the limiting formulas. These remainders were estimated in §6. The result is written as a series w.r.t. our basis and the corresponding terms are included in $\beta_{j,p}^{(u)}$, $\beta_{j',p}^{(n)}$ and the stable part of the spectrum.

Finally we have

$$b_{j,p+1}^{(u)} = b_{j,p}^{(u)} \left(1+\frac{1}{p}\right)^{\alpha_j^{(u)}}, \quad b_{j,p+1}^{(n)} = b_{j,p}^{(n)}$$

and the formulas for $\beta_{j,p}^{(u)}$, $\beta_{j',p}^{(n)}$ and $\Phi_p^{(st)}(Y,s)$. This works for $p < p_{m+1}$. If $p = p_{m+1}$, then we introduce new variables (rescaling!)

$$b_{j,p_{m+1}}^{(u)} = b_{j,p_{m+1}-1}^{(u)} \left(1+\frac{1}{p_{m+1}}\right)^{\alpha_j^{(u)}} + \beta_{j,p_{m+1}}^{(u)},$$
$$b_{j',p_{m+1}}^{(n)} = b_{j',p_{m+1}-1}^{(n)} + \beta_{j',p_{m+1}}^{(n)}.$$

It is our other inductive assumption that

$$-\rho_1^m \le b_{j,p_m}^{(u)} \le \rho_1^m, \quad -\rho_1^m \le b_{j,p_m}^{(n)} \le \rho_1^m$$

where $0 < \rho_1 < 1$ but ρ_1 is sufficiently close to 1.

Let $\Delta_{m+1}^{(m+1)} = \left[-\rho_1^{m+1}, \rho_1^{m+1}\right]$ and $\Delta_m^{(m+1)} = \left\{(b_{j,p_m}^{(u)}, b_{j',p_m}^{(n)}) :\ -\rho_1^{m+1} \le b_{j,p_{m+1}}^{(u)},\ b_{j',p_{m+1}}^{(n)} \le \rho_1^{m+1}\right\}$. It follows easily from the estimates of $\beta_{j,p_{m+1}}^{(u)}$, $\beta_{j',p_{m+1}}^{(n)}$ that $\Delta_m^{(m+1)} \subseteq \Delta_m^{(m)}$. If $\Delta_0^{(m)} = \left\{(b_j^{(u)}, b_{j'}^{(n)}) :\ (b_{j,m}^{(u)}, b_{j',m}^{(n)}) \in \Delta_m^{(m)}\right\}$, then $\Delta_0^{(m)}$ is a decreasing sequence of closed sets. The intersection $\bigcap_m \Delta_0^{(m)}$ gives us the values of parameters for which $\delta^{(p)} \to \infty$ as $p \to \infty$.

We make also some shortening of the time interval $S^{(m)}$. In the formulas for $\delta^{(r)}$ there are several remainders which appear when we replace in all expressions s' and s'' by s. We

estimate these remainders using the fact that our functions satisfy the Lipschitz condition and the Lipschitz constants and the maxima of their values decay as some power of p. We choose the interval $S^{(m+1)} \subset S^{(m)}$ so that when we consider $s \in S^{(m+1)}$ these remainders do not violate the basic inclusion $\Delta_m^{(m+1)} \subset \Delta_m^{(m)}$. It is easy to see $S^{(m+1)}$ can be chosen so that $S^{(m)} \setminus S^{(m+1)}$ consists of two intervals whose lengths decay exponentially. Therefore $\bigcap_m S^{(m)}$ is an interval of positive length.

The transformation $(b^{(u)}_{j,p_{m+1}}, b^{(n)}_{j',p_{m+1}}) \to (b^{(u)}_{j,p_m}, b^{(n)}_{j',p_m})$ is given by smooth functions and is sufficiently close to the identity map. The last step in the renormalization is the replacement in all $\delta^{(r)}$, $r < p_{m+1}$ the variables $b^{(u)}_{j,p_m}$, $b^{(n)}_{j',p_m}$ by their expressions through $b^{(u)}_{j,p_{m+1}}$, $b^{(n)}_{j',p_m+1}$. The form of $\delta^{(r)}$ in new variables is the same as before.

The Choice of Constants

The main constants which are used in the construction are the following:

1. $k^{(0)}$ which determines the position of the domain where $v(k, 0)$ is concentrated.
2. D_1 is the constant which determines the size of the neighborhood where $v(k, 0)$ is concentrated.
3. ρ_1 determines the size of the neighborhood where the main parameters $b^{(u)}_j$, $b^{(n)}_{j'}$ vary.
4. D_2 is the constant which determines the possible size of perturbations $\Phi^{(st)}$ in the form of $v(k, 0)$.
5. λ_1 is the power which gives the estimation of the decay of h_r in the domain B.
6. λ_2 is the parameter which determines the size of the first part of the procedure.
7. ϵ determines the values of p where we make the renormalization.

The value of $k^{(0)}$ should be sufficiently large. All estimate of the remainders which appear during the first half of the procedure are less than $\frac{const}{(k^{(0)})^{\frac{1}{2}}}$. They should be so small that the estimates of all $\beta^{(u)}_{jr}$, $\beta^{(n)}_{j'r}$ are much smaller than ρ_1. On the other hand, ρ_1 should be small but not too small. It should be small in order to make the quadratic part of our formulas smaller than the linear part. However ρ cannot be too small in order that we could choose

the next interval $[-\rho^{m+1}, \rho^{m+1}]$. This can be achieved by the choice of $k^{(0)}$. The parameter λ_2 should be small. In this case the estimates of all corrections are easier. However, after λ_2 is chosen the value of $k^{(0)}$ can be taken sufficiently large depending on λ_2. The parameter λ_1 can be arbitrarily large in order to make the perturbation arbitrarily small. The value of D_1 determines the estimates in the domain B which decay as $\frac{1}{\left(k^{(0)}\right)^{\lambda_1}}$. We choose D_1 so that $\lambda_1 > \frac{1}{2}$. The value of ϵ is chosen small so that we can write with a good precision the action of the linearized renormalization group.

§10. Critical Value of Parameters and Behavior of Solutions near the Singularity Point

We return back to the first formulas:

$$v_A(k,t) \;=\; \exp\{-t|k|^2\}\, A\cdot v(k,0) \,+\, \int_0^t \exp\{-(t-s)|k|^2\}\,\cdot\, \sum_{p>1} A^p h_p(k,s)\, ds$$

or

$$v_A(k,t) \;=\; \exp\{-\,t|k|^2\}\, A\cdot v(k,0) \,+\, \int_0^t \exp\{-\,(t-s)|k|^2\}\,\cdot\, \sum_{p>1} A^p g_p(k\sqrt{s},s)\, ds\,. \quad (41)$$

Our construction gives us the interval $S = \bigcap_n S^{(n)}$ on the time axis such that for each $t \in S$ we can find the values of parameters $b_j^{(u)} = b_j^{(u)}(t)$, $1 \le j \le 4$ and $b_{j'}^{(n)} = b_{j'}^{(n)}(t)$, $1 \le j' \le 6$ such that we have the representation (31) with $\delta^{(r)} \to 0$ as $r \to \infty$. It is easy to see that $A_{cr}(t) = \Lambda^{-1}(t)$. If so then $A^p h_p(k,t)$ is concentrated in the domain with the center at $\frac{\kappa^{(0)}p}{\sqrt{t}}$ having the size $O(\sqrt{p})$ and there it takes values $O(p)$. This immediately implies that at t the energy is infinite.

Consider $t' < t$. It is important to investigate the behavior of $E(t')$ and the enstrophy $\Omega(t')$ of the same solution with $A = A_{cr}(t)$ when t' is close to t. Denote $\Delta t = t - t'$. It follows easily from the proof of the main result that $\Lambda(t')/_{\Lambda(t)} = (1 - C_1\Delta t + O(\Delta t))$ for some constant C_1. Since $A_{cr}^p \cdot (\Lambda(t'))^p = A_{cr}^p \cdot (\Lambda(t))^p \cdot \left(\Lambda(t')/_{\Lambda(t)}\right)^p = (1 - C\Delta t + o(\Delta t))^p$. It is clear that the terms in (41) are close to each other for $p \le O\left(\frac{ln(\Delta t)^{-1}}{\Delta t}\right)$. For $p >> \frac{ln(\Delta t)^{-1}}{\Delta t}$ the

product $A_{cr}^p(\Lambda(t'))^p$ tends exponentially to zero and dominates other terms of the expansion. Therefore it is enough to consider $|k| \leq O\left(\frac{ln(\Delta t)^{-1}}{\Delta t}\right)$ and in this domain the solution grows as $|k|^{\frac{3}{2}}$. The factor $|k|^{\frac{1}{2}}$ appears because for any k the values for which the terms in (41) give the essential contribution to the solution belonging to an interval of the size $O(\sqrt{|k|}) = O(\sqrt{p})$. From this argument it follows easily that $E(t') \sim \left(\frac{ln(\Delta t)^{-1}}{\Delta t}\right)^6$ and $\Omega(t') \sim \left(\frac{ln(\Delta t)^{-1}}{\Delta t}\right)^8$.

Appendix I. Hermite Polynomials and their basic properties

Take $\sigma > 0$ and write

$$He_n^{(\sigma)}(x) = (-1)^n e^{\frac{\sigma x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{\sigma x^2}{2}}, \quad n \geq 0.$$

It is clear that $He_n^{(\sigma)}(x) = \sigma^n x^n + \cdots$, where dots mean terms of smaller degree. We shall call $He_n^{(\sigma)}$ the n-th Hermite polynomial. It is clear that $He_0^{(\sigma)}(x) = 1$, $He_1^{(\sigma)}(x) = \sigma x$, $He_2^{(\sigma)}(x) = \sigma^2 x^2 - \sigma$ and so on. In general, $He_n^{(\sigma)}(x) = \sigma^{\frac{n}{2}} He_n^{(1)}(\sqrt{\sigma} x)$. It is easy to check that

$$\sigma x He_n^{(\sigma)}(x) = He_{n+1}^{(\sigma)}(x) + \sigma n He_{n-1}^{(\sigma)}(x). \tag{42}$$

The Fourier transform of $He_m^{(\sigma)}(x) e^{-\frac{\sigma x^2}{2}} \sqrt{\frac{\sigma}{2\pi}}$ is $(i\lambda)^m e^{-\frac{\lambda^2}{2\lambda}}$. This implies the formula for convolution:

$$\int_{\mathbb{R}^1} He_{m_1}^{(\sigma)}(x-y) e^{-\frac{\sigma(x-y)^2}{2}} \sqrt{\frac{\sigma}{2\pi}} \cdot He_{m_2}^{(\sigma)}(y) e^{-\frac{\sigma y^2}{2}} \sqrt{\frac{\sigma}{2\pi}} dy = He_{m_1+m_2}^{(\sigma)}(x) e^{-\frac{\sigma x^2}{2}} \sqrt{\frac{\sigma}{2\pi}} \tag{43}$$

Take positive γ_1, γ_2, $\gamma_1 + \gamma_2 = 1$ and consider the convolution of $He_{m_1}^{(\sigma)}(\frac{x}{\sqrt{\gamma_1}}) e^{-\frac{\sigma x^2}{2\gamma_1}} \cdot \sqrt{\frac{\sigma}{2\pi\gamma_1}}$ and $He_{m_2}^{(\sigma)}(\frac{x}{\sqrt{\gamma_2}}) e^{-\frac{\sigma x^2}{2\gamma_2}} \cdot \sqrt{\frac{\sigma}{2\pi\gamma_2}}$. Their Fourier transforms are $(i\lambda\sqrt{\gamma_1})^{m_1} e^{-\frac{\lambda^2\gamma_1}{2\sigma}}$ and $(i\lambda\sqrt{\gamma_2})^{m_2} e^{-\frac{\lambda^2\gamma_2}{2\sigma}}$ respectively. The product of these two functions is $\gamma_1^{\frac{m_1}{2}} \gamma_2^{\frac{m_2}{2}} (i\lambda)^{m_1+m_2} e^{-\frac{\lambda^2}{2\sigma}}$. Therefore the convolution is $\gamma_1^{\frac{m_1}{2}} \cdot \gamma_2^{\frac{m_2}{2}} He_{m_1+m_2}^{(\sigma)}(x) e^{-\frac{\sigma x^2}{2}}$.

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